# ON SETS WHICH MEET EACH LINE IN EXACTLY TWO POINTS

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Abstract

ABSTRACT. Using techniques from geometric measure theory and descriptive set theory, we prove a general result concerning sets in the plane which meet each straight line in exactly two points. As an application we show that no such "two point" set can be expressed as the union of countably many rectifiable sets together with a set with Hausdorff 1-measure zero. Also, as another corollary, we show that no analytic set can be a two point set provided every purely unrectifiable set meets some line in at least three points. Some generalizations are given to "n-point" sets and some other geometric constructions.

To the memory of Paul Erdös

## 1. *Introduction*

In 1914, Mazurkiewicz [**7**] showed that there is a "two-point" subset $M$ of $\mathbf{R}^2$, *i.e.*, $M$ meets each line in exactly 2 points. One can easily modify Mazurkiewicz's argument to show that for each positive integer $n$, $n \geq 2$, there is an "$n$-point" subset $M$ of $\mathbf{R}^2$; a set $M$ which meets each line in exactly $n$ points. More refined generalizations of this result were given by Erdös and Bagemihl [**1**]. The axiom of choice played a central role in these constructions. There is one indication that perhaps the axiom of choice is not needed. Consider the set $M$ which is the union of all circles with center the origin and radius a positive integer. This $F_\sigma$ set meets every line in a countably infinite set. Thus, the question naturally arises as to how effective a construction of an $n$-point set can be. Specifically, can a two point set be a Borel set? This question has been known for many years. I believe I first heard the problem from Erdös, who said it had been around since he was a "baby." Larman showed that if there is such a Borel set, then it must be somewhat complex [**5**]. He showed that a 2 point set cannot be an $F_\sigma$ set. Let me mention that it is also known that if $M$ is analytic and $M$ is an $n$ point set, then $M$ is a Borel set. This follows from the fact that every analytic subset A of $\mathbf{R}^2$ such that each vertical fiber $A_x$ has cardinality $\leq n$ lies in a Borel set $B$ such that each vertical fiber has cardinality $\leq n$. Also, Miller has shown that if one assumes Gödel's axiom of constructibility, $V = L$, then there is a 2 point set which is a coanalytic set [**8**]. It is also known that a two point set must have topological dimension zero [**4**]. I have discussed this in problem 1069 in [**9**]. We will prove a theorem in this paper which implies that a two point set cannot be a $\sigma$- rectifiable set. An old unsolved problem in geometric

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measure theory(see [**6**] p. 258) is whether the following proposition is true:

(P2) every purely unrectifiable compact 1-set in $\mathbf{R}^2$ must meet some line in at least three points or,
more generally,
(Pn) every purely unrectifiable compact 1-set in $\mathbf{R}^2$ must meet some line in at least $n+1$ points.

Among several results, we will prove the following two theorems.

THEOREM 9. *Let $n$ be an integer, $n \geq 2$. Let $M$ be a subset of $\mathbf{R}^2$ which meets every line in exactly $n$ points. Then $M$ is not a $\sigma$-rectifiable 1-set.*

THEOREM 11. *Suppose $n \geq 2$ and proposition (Pn) is true. Then there is no analytic subset of $\mathbf{R}^2$ which meets every straight line in exactly $n$ points.*

## 2. *Results*

First, let us recall some facts and prove some elementary auxiliary theorems and lemmas. One fundamental property of Hausdorff measures which will be used here is the fact that if an analytic set A has positive $s$-dimensional Hausdorff measure, $\mathcal{H}^s(A) > 0$, then there is a compact set $K \subset A$ such that $0 < \mathcal{H}^s(K) < \infty$. This theorem is proved in some form in several books [**2**], [**6**],[**10**]. A new more general proof has been given by Howroyd [**3**]. The open ball with center $x$ and radius $r$ is denoted by $B(x, r)$. Lebesgue measure is denoted by $\lambda$. We begin with some simple facts.

**Lemma 1** *If $A \subset \mathbf{R}^2$ is analytic and at most countably many lines meet $A$ in an uncountable set, then the Hausdorff dimension of $A$,* $\dim_H(A) \leq 1$.

**Proof.** Suppose $\dim_H(A) > s > 1$. Since A is analytic, A would contain a compact set $E$ with $0 < \mathcal{H}^s(E) < \infty$ [**10**]. By a theorem of Marstand (see [**2**], p. 93), there would be a point $x$ of $E$ and uncountably many lines $L$ through $x$ such that $\dim_H(E \cap L) = s - 1 > 0$. So, there would be uncountably many lines which meet A in an uncountable set, a contradiction. ■

**Lemma 2.** *If $W \subset \mathbf{R}^2$ and the orthogonal projection of $W$ onto some line has positive $\mathcal{H}^1$ measure, then* $\dim_H(W) \geq 1$. *In particular, if $W$ meets every line parallel to some fixed line in a nonempty set, then $\mathcal{H}^1(W) = \infty$.*

**Proof.** Consider $P$, the orthogonal projection of $W$ onto a line $L$. Since $P$ is nonexpansive, $0 < \mathcal{H}^1(P(W)) \leq \mathcal{H}^1(W)$. This means that $\dim(M) \geq 1$. If $P(W) = L$, then for the same reason, $\mathcal{H}^1(W) = \infty$. ■

**Corollary 3.** *If $A$ is an analytic set in $\mathbf{R}^2$ and $A$ meets every line in a countable nonempty set, then* $\dim_H(A) = 1$ *and* $\mathcal{H}^1(A) = \infty$.

**Remark.** One can construct a two point set $H$ which has positive measure with respect to Lebesgue measure. Of course, such a set $H$ is not Lebesgue measurable.

We recall several more facts. A set $S \subset \mathbf{R}^2$ is a 1-set means $0 < \mathcal{H}^1(S) < \infty$.

Also, every analytic set A in $\mathbf{R}^2$ with positive $\mathcal{H}^1$ measure contains a 1-set, [Fal, Rog] and every 1-set can be expressed as the disjoint union of its rectifiable and purely unrectiable parts, (see [**2**], *p.* 26). There are several different equivalent ways of defining rectifiable. For our purposes, a 1-set $E$ is rectifiable means there is a sequence, $\{\Gamma_i\}_{i=1}^{\infty}$, of $C^1$ arcs such that $\mathcal{H}^1(E\setminus\cup\Gamma_i)=0$. We can also assume these arcs have a tangent at each its points. A 1-set $E$ is purely 1-unrectifiable means for every $C^1$ arc $\Gamma$, $\mathcal{H}^1(E\cap\Gamma)=0$ (see [**2**] [**6**]). A set E is $\sigma$-rectifiable means it can be expressed as the union of countably many rectifiable 1-sets.

**Theorem 4.** *Let $T$ be a Borel subset of $R$ and let $f:T\to R$ be a Borel measurable function such that $G=Gr(f)$, the graph of $f$ is $\sigma$-rectifiable. Then there is a sequence of pairs, $\{([a_i,b_i],g_i)\}_{i=1}^{\infty}$ such that for each $i$, $g_i:[a_i,b_i]\to R$ is of class $C^1$ and $\lambda(proj_1(G\setminus\cup Gr(g_i)))=0$.*

**Proof.** We can express $G$ as $G=\cup_i E_i$, where each $E_i$ is a rectifiable 1-set. It suffices to show that each $E_i$ can be covered as indicated in the conclusion. Towards this end, fix $i$, set $E=E_i$ and let $h_j:[0,1]\to\mathbf{R}^2$, $j=1,2,3...$ be one-to-one $C^1$ maps such that $\mathcal{H}^1(E\setminus\cup\Gamma_j)=0$, where $\Gamma_j$ is the image of $h_j$.

Fix $j$. Let $\Gamma=\Gamma_j$, let $h=h_j=(\varphi_1,\varphi_2)$ and let $L$ be the length of the rectifiable arc $\Gamma$. It suffices to show the conclusion holds for $E\cap\Gamma$. Let $V=\{x:(x,f(x))\in E\cap\Gamma \text{ and } \Gamma \text{ has a vertical tangent at } (x,f(x))\}$. We claim $\lambda(V)=0$. By way of contradiction, suppose $\lambda(V)>0$. Let $M=\max(1,3L/\lambda(V))$. Let $\mathcal{V}=\{[x-\epsilon_x,x+\epsilon_x]:x\in V,\ \Gamma\cap B((x,f(x)),\sqrt{1+M^2}\epsilon_x)\subset C(x,M)$ and $\Gamma\cap S(x,\sqrt{1+M^2}\epsilon_x)\neq\emptyset\}$, where $C(x,M)$ is the cone with vertex $(x,f(x))$, axis the vertical line through $(x,f(x))$, and boundary the lines through the vertex with slopes $\pm M$ and $S(x,\sqrt{1+M^2}\epsilon_x)$ is the circle with center $(x,f(x))$ and radius $\sqrt{1+M^2}\epsilon_x$. Since $\mathcal{V}$ is a Vitali cover of $V$, there are pairwise disjoint intervals $I_i=[x_i-\epsilon_i,x+\epsilon_i]\in\mathcal{V}$ such that $\sum\epsilon_i>\lambda(V)/3$. For each $i$, let $t_i$ be the number such that $h(t_i)=(x_i,f(x_i))$ and let $s_i$ be such that $h(s_i)\in S(x_i,\sqrt{1+M^2}\epsilon_i)$ and $h$ maps the open interval from $s_i$ to $t_i$ into $C(x_i,M)\cap B((x_i,f(x_i)),\sqrt{1+M^2}\epsilon_i)$. Since the intervals $[s_i,t_i]$ are pairwise disjoint, $L\geq\sum_i\|h(s_i)-h(t_i)\|\geq\sum_i M\epsilon_i\geq M(\lambda(V)/3)>L$. This is a contradiction and establishes the claim.

For each $x\in\text{proj}_1(\Gamma\cap E)\setminus V$, let $t_x=h^{-1}((x,f(x)))$ and let $\delta_x>0$ be such that if $|s-t_x|\leq\delta_x$, then the slope of the tangent line to $\Gamma$ at $h(s)$ is in absolute value $<|M_x|+1$, where $M_x$ is the slope of the tangent line to $\Gamma$ at $h(t_x)$. In particular, $\varphi_1'(s)\neq 0$, for $s\in[t_x-\delta_x,t_x+\delta_x]=J_x$. Let $[a_x,b_x]=\varphi_1(J_x)$. Then $\varphi_1$ is a $C^1$ homeomorphism of these two intervals. Define $g_x:[a_x,b_x]$ by $g_x(u)=\varphi_2(\varphi_1^{-1}(u))$. By taking a suitable countable collection of these intervals and maps, we obtain the required sequence of pairs. ■

For the next two theorems, let us suppose we have fixed a cartesian coordinate system. For each point $(u,v)\in\mathbf{R}^2$ with $v\neq 0$, let $T_{(u,v)}$ be the projection from $(u,v)$ onto the $x$-axis; *i.e.*, for each point $(x,y)$ with $y\neq v$, $T_{(u,v)}((x,y))=z$, where $(x,y)$, $(u,v)$ and $(z,0)$ are collinear. To say that $C$ is a positive cone means there is a point $(0,w)$ and an angle $\theta$, with $0<\theta<\pi/2$ such that $C$ consist of all points $p\in\mathbf{R}^2$ such that the angle between the vector $p-(0,w)$ and the positive $y$-axis is less than $\theta$.

**Theorem 5.** *Suppose $f:[a,b]\to R$ is $C^1$ and $|f'(x)|\leq M$, for $x\in[a,b]$,*

*$E \subset [a,b]$ is a Borel set and $\lambda(E) > 0$. Let $G = Gr(f)$. For each $\tau$, $0 < \tau < 1$, there is a positive cone $C$ such that if $(u,v) \in C$, then $\lambda(T_{(u,v)}(G \cap E \times R) \cap [a,b]) > \tau\lambda(E)$.*

**Proof.** Let $l_1$ be the line of slope $M$ which intersects $G$ and $G$ lies in the closed lower half plane determined by $l_1$. Similarly, let $l_2$ be the line of slope $-M$ which intersects $G$ and $G$ lies in the closed lower half plane determined by $l_2$ and let $W$ be the cone defined by $l_1$ and $l_2$ which lies opposite G. If $(u,v) \in W$, then $T_{(u,v)}$ is one-to-one on $G$ and $T_{(u,v)}((x, f(x)) = x - f(x)\dfrac{u-x}{v-f(x)}$. Define $g = g_{(u,v)} : [a,b] \to R$ by $g(x) = T_{(u,v)}(x, f(x))$. So,

$$g'(x) = 1 - \frac{f'(x)(u-x)}{v-f(x)} + \frac{f(x)}{v-f(x)} - \frac{f(x)f'(x)(u-x)}{(v-f(x))^2}.$$

We can make the derivatives of $g$ and of $g^{-1}$ uniformly as close to one as we wish by taking v large enough and $|u/v|$ small enough. Therefore, there is a positive cone $C$ such that if $(u,v) \in C$, then $g_{(u,v)}$ is a continuous one-to-one map of $[a,b]$ onto a subinterval of $[a - (1-\tau)\lambda(E)/4, b + (1-\tau)\lambda(E)/4]$ and

$$\lambda(g_{(u,v)}(E)) = \lambda(T_{(u,v)}(G \cap (E \times R))) > (\frac{1+\tau}{2})\lambda(E).$$

Thus, for $(u,v) \in C$, we have $\lambda(T_{(u,v)}(G \cap (E \times R))) \cap [a,b]) > \tau\lambda(E)$. ■

**Theorem 6.** *Suppose $f_i : [c,d] \to R, i = 1, ..., n$ are $C^1$, $E \subset [c,d]$ is a Borel set and $\lambda(E) > 0$. Let $G_i = Gr(f_i), i = 1, ..., n$. There is a positive cone $C$ such that if $(u,v) \in C$, then some line through (u,v) meets each set $G_i$ in points whose first coordinates are in E.*

**Proof.** Let $x$ be a point of $E$ which is a density point of $E$. Choose an interval $[a,b]$ centered at $x$ such that $\lambda(E \cap [a,b]) > (\dfrac{n-1}{n})(b-a)$. It follows from theorem 5 that there is a positive cone $C$ such that if $(u,v) \in C$, then $\lambda(T_{(u,v)}(G_i \cap (E \times R)) \cap [a,b]) > (\dfrac{n-1}{n})(b-a)$, for $i = 1, ..., n$. Thus, if $(u,v) \in C$, then $\cap_{i=1}^{n} T_{(u,v)}(G_i \cap (E \times R)) \cap [a,b] \neq \emptyset$. ■

**Theorem 7.** *Let $n$ be a positive integer, $n \geq 2$. Suppose $M \subset \mathbf{R}^2$ is such that for every direction $\theta$ and cartesian coordinate system with x-axis in the direction $\theta$, there is an interval $[a,b]$, $C^1$ functions $f_i : [a,b] \to R, i = 1, ..., n$, a Borel subset $E$ of $[a,b]$ with $\lambda(E) > 0$ such that the graphs of the functions $f_i; i = 1, ..., n$ over the set $E$ are pairwise disjoint subsets of M. Then either $M$ is bounded or else some line meets $M$ in at least $n+1$ points.*

**Proof.** Suppose no line meets $M$ in at least $n+1$ points. For each vector $u$ with $\|u\| = 1$, it follows from theorem 6 that there is some $r_u > 1$ and $\pi/4 > \theta_u > 0$ such that no points of $M$ lie in the cone, $C_u$, with vertex $r_u u$ and consisting of all points $z$ such that the angle between $z - r_u u$ and the positive ray determined by $u$ is less than $\theta_u$. For each $u$, let $A_u$ be the open arc subtended on the circle $|r| = r_u + 1$ by the boundary rays of the cone $C_u$. Let $I_u$ be the central projection of $A_u$ onto the unit circle. Let $I_{u_0}, ..., I_{u_{m-1}}$ cover the unit circle and suppose we have enumerated these arcs in a counterclockwise manner. For each $i$, let $h_i^-$ and $h_i^+$ be the right and left rays with vertex the origin determined by the endpoints of $I_{u_i}$. Since the unbounded region with boundary formed by the arc $A_{u_i}$ and the rays $h_i^-$ and $h_i^+$ is a subset of the cone $C_u$, there are no points of M in it. Let $R = \max_i\{r_{u_i} + 1\}$. Then there is no point, $p \in M$ with $\|p\| > R$. ∎

In the next theorem, $M_x$, the $x$-section of a subset $M$ of the plane is given by $M_x = \{y : (x, y) \in M\}$.

**Theorem 8.** *Suppose $M \subset \mathbf{R^2}$ is $\mathcal{H}^1$-measurable, $\sigma$-rectifiable, and $A = \{x : card(M_x) \geq n\}$ has positive Lebesgue measure. Then there is an interval [c,d], and $C^1$ functions, $f_i : [c, d] \to \mathbf{R}$, i=1,...,n and a Borel subset E of [c,d] with $\lambda(E) > 0$ such that the graphs of the functions $f_i$ over E are pairwise disjoint subsets of M.*

**Proof.** There is a Borel set $T \subset A$ with $\lambda(T) > 0$ and Borel measurable functions $h_i; i = 1, ..., n$ such that the graphs, $G_i$, of the functions $h_i$ are pairwise disjoint subsets of M. Since each set $G_i$ is $\sigma$-rectifiable, by theorem 4, there are $C^1$ functions $g_{ij} : [a_{ij}, b_{ij}] \to \mathbf{R}$ such that

$$\lambda(proj_1(G_i \setminus \cup_j(Gr(g_{ij}))) = 0.$$

Let $E_{ij} = [a_{ij}, b_{ij}] \cap T$. For each i, $\lambda(T \setminus \cup_j E_{ij}) = 0$. Choose $x \in T$ and $j_1, ..., j_n$ such that $x$ is a density point of each $E_{ij_i}$. Now, choose an interval $[a, b] \subset \cap_i[a_{ij_i}, b_{ij_i}]$ with $x$ in its interior such that $\lambda(E) > 0$, where $E = \cap_i E_{ij_i} \cap [a, b]$. Let $f_i$ be $g_{ij_i}$ restricted to [a,b]. ∎

As an immediate corollary of theorems 7 and 8 we have:

**Theorem 9.** *Let n be an integer, $n \geq 2$. Let $M$ be an $\mathcal{H}^1$ measurable subset of $\mathbf{R}^2$ which meets every line in exactly n points. Then $M$ is not $\sigma$-rectifiable.*

**Proposition Pn and n-point sets.** In the next two theorems, we assume that $n \geq 2$ and every purely unrectifiable compact 1-set meets some line in at least $n + 1$ points. Since it is not known whether this proposition is true, these results are somewhat tentative.

**Theorem 10.** *Assume $n \geq 2$ and proposition (Pn) is true. Let A be an analytic subset of $\mathbb{R}^2$. Suppose $\mathcal{H}^1(A) > 0$ and at most countably many lines meets A in at least $n + 1$ points, then $A = \cup E_i \cup N$, where each $E_i$ is a compact rectifiable 1-set and $\mathcal{H}^1(N) = 0$.*

**Proof.** Suppose A were to include a compact purely unrectifiable 1-set W. We could also assume that W misses the countably many lines which meet A in at least $n + 1$ points. But, since proposition (Pn) holds, W and therefore, A, would meet some additional line in at least $n + 1$ points. Thus, every compact 1-set lying in A is rectifiable. If the final conclusion were not true, then, by transfinite induction, A would contain uncountably many pairwise disjoint compact 1-sets, $E_\alpha, \alpha < \omega_1$. Since each $E_\alpha$ is rectifiable, $\lambda(proj_\theta(E_\alpha)) > 0$, for all but at most one value of $\theta$, where $proj_\theta(E)$ means the orthogonal projection of E onto the line $L_\theta$ through the origin that makes angle $\theta$ with the x-axis and $\lambda$ is Lebesgue measure(or $\mathcal{H}^1$ measure) on the line $L_\theta$ (see [**2**],p. 84). So, there is some $\theta$ and some $c > 0$ such that for uncountably many $\alpha$, $\lambda(proj_\theta(E_\alpha)) \geq c$. This means there is some $x$ on $L_\theta$ with

$x \in proj_\theta(E_\alpha)$, for uncountably many $\alpha$. Thus, the line through $x$ perpendicular to $L_\theta$ meets A in uncountably many points. This is a contradiction. ■

As a corollary of theorems 9 and 10, we have

**Theorem 11.** *Suppose $n \geq 2$ and proposition (Pn) is true. Then there is no analytic subset of $\mathbf{R}^2$ which meets every straight line in exactly n points.*

**SOME GENERALIZATIONS.** Let $\mathcal{L}$ be the space of all lines and let $q : \mathcal{L} \to \omega \cup \{\omega, 2^\omega\}$. In [B-E], Bagemihl and Erdös, using the axiom of choice show that if $q(L) \geq 2$, for each L, then there is a set $E$ such that for each line $L$, $card(E \cap L) = q(L)$.

**QUESTION.** Suppose $q : \mathcal{L} \to \mathbb{N}$ is such that for each line L, $q(L) \geq 2$. Under what conditions is there a Borel set B such that for each line L, $card(B \cap L) = q(L)$?

Theorem 11 shows that if q has the constant value n and proposition (Pn) is true, then there is no such Borel set. As the next theorem shows, there is one necessary condition: the function q must be Borel measurable.

**Theorem 12.** *Let $B \subset \mathbf{R}^2$ be a Borel set and suppose for each line L, $q(L) = \mathrm{card}(B \cap L)$. Then $q : \mathcal{L} \to \omega \cup \{\omega, 2^\omega\}$ is measurable with respect to $\mathcal{B}(\mathcal{A})$, the $\sigma$-algebra generated by the analytic sets. Moreover, if, for each L, $1 \leq q(L) < \omega$, then the function q is Borel measurable.*

**Proof.** Let $g : F \to B$ be a continuous one-to-one map of the closed subset F of $\omega^\omega$ onto B. Let $q : \mathcal{L} \to \mathcal{R}$ be given by $q(L) = \mathrm{card}(L \cap B)$. Then, for $2 \leq n \leq \omega$,

$$q(L) \geq n \Leftrightarrow \exists x_1, \ldots, x_n [x_i \neq x_j,\ i \neq j \text{ and } \forall i f(x_i) \in L]$$

and

$$q(L) = 2^\omega \Leftrightarrow \exists P [P \subset F,\ P \text{ is perfect},\ f(P) \subset L].$$

It follows from these equivalences that $q$ is measurable with respect to $\mathcal{B}(\mathcal{A})$. If, for each $L$, $2 \leq q(L) < \omega$, consider the map $G : \mathcal{L} \to \mathcal{K}(\mathbf{R}^2)$, the space of compact subsets of $\mathbf{R}^2$, given by $G(L) = L \cap B$. Let

$$H_n = \{K \in \mathbf{R}^2 : \mathrm{card} K = n\}.$$

For each $n$, $q^{-1}(n) = G^{-1}(H_n)$. The function $G$ is Borel measurable. To see this consider

$$W = \{(L, x) : x \in L \cap B\}.$$

$W$ is a Borel subset of $\mathcal{L} \times \mathbf{R}^2$ and for each $L$, $W_L$ being finite, is compact. Thus, the map $\mathcal{L} \to \mathcal{W}_\mathcal{L} = \mathcal{G}(\mathcal{L})$ is Borel measurable. ■

Of course, as the next easy example shows it can happen that a given function $q$ has a "Borel" realization. The combinatorics of when such a Borel set exist seem quite complicated.

**Example.** Let $B = \{(x, y) : y = \pm\frac{1}{x}\} \cup \{(x, y) : x^2 + y^2 = 2\}$. Then $B$ is a closed subset of $\mathbf{R}^2$ and $q(L)$ for each line $L$, $2 \geq \mathrm{card}(L \cap B) \leq 6$.

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