On the convergence of $\Sigma_{n=1}^{\infty} f(nx)$ for measurable functions

by

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ABSTRACT. We answer questions of Haight and of Weizsäcker by proving the following theorem: **Theorem 1.** *There exists a measurable function* $f:(0,+\infty)\to\{0,1\}$ *and two nonempty intervals* $I_F,\ I_\infty \subset [\frac{1}{2},1)$ *such that for every* $x\in I_\infty$ *we have* $\sum_{n=1}^{\infty} f(nx) = +\infty$ *and for almost every* $x\in I_F$ *we have* $\sum_{n=1}^{\infty} f(nx) < +\infty$. The function f is the characteristic function of an open set E.

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INTRODUCTION

Recently one of us was reminded of a problem of Weizsäcker [W]: Given a Lebesgue measurable function $f : \mathbf{R}^+ \to \mathbf{R}^+$, is it true that either for Lebesgue measure almost every $x > 0$, the series $\Sigma_{n=1}^{\infty} f(nx)$ converges or, else for Lebesgue measure almost every $x > 0$, the series $\Sigma_{n=1}^{\infty} f(nx)$ diverges? Weizsäcker investigated this problem in his Diplomarbeit and in particular showed that if the function f is in L_1, then the series converges almost surely. At about the same time, Haight [H1] showed that there is a Lebesgue measurable subset E of the positive real line with infinite measure such that if t and s are two distinct numbers in E then $t/s \notin \mathbf{N}$ and for each positive x, there are only finitely many positive integers n such that $nx \in E$. Thus, letting f be the characteristic function of E, we have a measurable function which is not integrable and yet the series $\Sigma_{n=1}^{\infty} f(nx)$ converges for all $x > 0$. Haight generalized his construction in [H2] and reiterated his question: If E is a Lebesgue measurable subset of the positive real line with infinite measure and $N(x, E) = card\{n \in \mathbf{N} \mid nx \in E\}$, is it true that either $N(x, E) = \infty$ for almost all x or else $N(x, E) < \infty$ for almost all x? In this note we shall construct an open set E which shows that the answer to both questions is no.

DEFINITIONS AND NOTATION

For $x \in [1, \infty)$ we set $\Phi(x) = [\frac{1}{2}, 1) \cap \{\frac{x}{n} : n \in \mathbf{N}\}$.

The intervals used in the Theorem will be defined as $I_F = (\frac{8}{9}, 1)$ and $I_\infty = [\frac{16}{25}, \frac{16}{24}]$.

For $\beta \neq 0$ we set $||x||_\beta = \min\{|x - n\beta| : n \in \mathbf{Z}\}$. If $\beta = 1$, we simply write $||x||$ instead of $||x||_1$. Observe that $||-x||_\beta = ||x||_\beta$, $||x+y||_\beta \leq ||x||_\beta + ||y||_\beta$ and $||x|| = q||\frac{x}{q}||_{1/q}$ when $q > 0$.

The Lebesgue measure of the set A is denoted by $|A|$. We denote by $\chi_A(x)$ the characteristic function of A, that is, $\chi_A(x) = 1$ for $x \in A$, and $\chi_A(x) = 0$ otherwise.

In this paper we denote by $\log x$ the *logarithm in base* 2.

PRELIMINARY RESULT

We will use Kronecker's Theorem on simultaneous inhomogenous approximation [C, p. 53]. Here we state a special case of it which will be used later.

Kronecker's Theorem. *Assume $\theta_1, ..., \theta_L \in \mathbf{R}$ and $(\alpha_1, ..., \alpha_L)$ is a real vector. The following two statements are equivalent:*

A) For every $\epsilon > 0$, there exists $p \in \mathbf{Z}$ such that

$$||\theta_j p - \alpha_j|| < \epsilon, \text{ for } 1 \leq j \leq L.$$

B) If $(u_1, ..., u_L)$ *is a vector consisting of integers and*

$$u_1\theta_1 + ... + u_L\theta_L \in \mathbf{Z},$$

then

$$u_1\alpha_1 + ... + u_L\alpha_L \in \mathbf{Z}.$$

MAIN RESULT

Theorem 1 easily follows from the following Lemma.

Lemma. *There exists* $K_0 \in \mathbf{N}$ *such that for every* $k \geq K_0$*, there exists* N_k *with the property that for each integer* $\nu \geq N_k$*, there is an open set* $H_k \subset (2^{\nu-1}, 2^\nu)$ *for which* $I_\infty \subset \Phi(H_k)$ *and* $|I_F \cap \Phi(H_k)| < 5 \cdot 2^{-k}$.

Proof of Theorem 1 based on the Lemma. Using the Lemma, choose a sequence of integers $\nu_{K_0} < \nu_{K_0+1} < ...$ such that for each ν_k, $(k \geq K_0)$, there exists an H_k satisfying the conclusions of the Lemma for $\nu = \nu_k$.

Let $f(x) = \sum_{k=K_0}^\infty \chi_{H_k}(x)$. It is clear that for every $x \in I_\infty$ and for each $k = K_0, K_0+1, ...$ there exists n_k such that $n_k x \in H_k$. Since the sets H_k are pairwise disjoint, $n_k \neq n_{k'}$, if $k \neq k'$ and therefore $\sum_{n=1}^\infty f(nx) = \infty$ on I_∞. On the other hand, by the Borel-Cantelli Lemma for almost every $x \in I_F$, there exists K_x such that $nx \notin H_k$ for all $k \geq K_x$ and $n \in \mathbf{N}$. Hence, $\sum_{n=1}^\infty f(nx)$ is finite almost everywhere on I_F. This completes the proof of Theorem 1. ■

Proof of the Lemma. Fix k. It is clear from the Prime Number Theorem that there is a positive integer $N_k \geq 3$ such that if $\nu \geq N_k$, then there are 2^k primes $p_1, ..., p_{2^k}$ with

$$\frac{23}{16}2^\nu < p_1 < ... < p_{2^k} < \frac{24}{16}2^\nu.$$

For each $\nu \geq N_k$, set $L = 2^k + 2^{\nu-2} + 1$ and define α_j and θ_j as follows:

$$\alpha_j = \begin{cases} \frac{j}{2^k} & j = 1, ..., 2^k, \\ 0 & 2^k < j \leq L; \end{cases}$$

and

$$\theta_j = \begin{cases} \log p_j & j = 1, ..., 2^k, \\ \log n_j & 2^k < j \leq L, \end{cases}$$

where $n_j = 7/8 \cdot 2^\nu + j - 2^k - 1$, for $2^k < j \leq L$. We note that n_j runs through all the $2^{\nu-2} + 1$ integers beginning with $7/8 \cdot 2^\nu$ and ending with $9/8 \cdot 2^\nu$. We show that condition B of Kronecker's theorem holds. Indeed, if a vector $(u_1, ..., u_L)$ consists of integers and $u_1\theta_1 + ... + u_L\theta_L = t \in \mathbf{Z}$, then $p_1^{u_1} \cdots p_{2^k}^{u_{2^k}} \cdot \prod_{2^k < j \leq L} n_j^{u_j} = 2^t$. Note that if $1 \le j \le 2^k < j' \le L$, then $p_j > \frac{23}{16}2^\nu > \frac{9}{8}2^\nu \ge n_{j'}$. It follows from the Fundamental Theorem of Arithmetic that $u_j = 0$ for all $j, 1 \le j \le 2^k$. Since $\alpha_j = 0$ for $2^k < j \le L$, we have

$$u_1\alpha_1 + ... + u_L\alpha_L = 0 \in \mathbf{Z}.$$

This shows that Condition B of Kronecker's Theorem holds and hence Condition A is also true. Thus, for $\epsilon = \frac{1}{4\cdot 2^k}$, we can choose $q \in \mathbf{Z}$ such that

$$||\theta_j q - \alpha_j|| < \epsilon \text{ holds for all } j \le L.$$

The choice of ϵ and $\alpha_j = \frac{j}{2^k}$ for $j \le 2^k$ implies that $q \ne 0$. If $q > 0$, set $q' = q$ and $\alpha'_j = \alpha_j$. If $q < 0$, set $q' = -q$ and $\alpha'_j = 1 - \alpha_j$. Then in both cases $||\theta_j q' - \alpha'_j|| < \epsilon$ holds for $j \in \mathcal{I}$. Observe that in both cases the set $\{\alpha'_j : j \le 2^k\}$ equals (modulo 1) the set $\{\frac{j}{2^k} : j = 1, ..., 2^k\}$, and for $2^k < j \le L$, $0 = \alpha_j$ equals (modulo 1) $\alpha'_j = 1$. Since these are the only properties we use, we can assume without limiting generality, that $q > 0$ and in the sequel we use q and α_j instead of q' and α'_j. Dividing by q we find that

$$||\theta_j - \frac{\alpha_j}{q}||_{\frac{1}{q}} < \frac{\epsilon}{q} \text{ holds for } j \le L.$$

This means that if $j \le 2^k$, we have

$$||\log p_j - \frac{j}{q\cdot 2^k}||_{\frac{1}{q}} < \frac{1}{q\cdot 4\cdot 2^k},$$

while for $2^k < j \le L$, we have

$$||\log n_j||_{\frac{1}{q}} < \frac{1}{q\cdot 4\cdot 2^k}.$$

Set

$$G = \{x \in (\log\frac{8}{9} + \nu, \nu) : ||x||_{\frac{1}{q}} < \frac{1}{q\cdot 2^k}\}.$$

Clearly, G is open as is $H_k \subset (2^{\nu-1}, 2^\nu)$ which is defined by $H_k = \{2^x : x \in G\}$.

We next show $I_\infty \subset \Phi(H_k)$. Let $y \in I_\infty$ and let $x = \log y$. Since $\{\alpha_j : j \le 2^k\}$ equals (modulo 1) the set $\{\frac{j}{2^k} : j = 1, ..., 2^k\}$, we can choose $j_x \le 2^k$ such that

$$||x + \frac{j_x}{q\cdot 2^k}||_{\frac{1}{q}} \le \frac{1}{q\cdot 2\cdot 2^k}.$$

Then

$$||x + \log p_{j_x}||_{\frac{1}{q}} \le ||x + \frac{j_x}{q\cdot 2^k}||_{\frac{1}{q}} + ||\log p_{j_x} - \frac{j_x}{q\cdot 2^k}||_{\frac{1}{q}} \le \frac{1}{q\cdot 2\cdot 2^k} + \frac{1}{q\cdot 4\cdot 2^k} < \frac{1}{q\cdot 2^k}.$$

Since $\log\frac{16}{25}\le x\le \log\frac{16}{24}$ and $\log(\frac{23}{16}2^{\nu})<\log p_{j_x}<\log(\frac{24}{16}2^{\nu})$, we obtain

$$\nu+\log\frac{8}{9}<\nu+\log\frac{23}{25}\le x+\log p_{j_x}<\nu.$$

Thus, $x+\log p_{j_x}\in G$ which means $p_{j_x}y\in H_k$. Hence, $I_\infty\subset\Phi(H_k)$.

Finally, we show $|I_F\cap\Phi(H_k)|<5\cdot 2^{-k}$, if k is sufficiently large. Towards this end, set $G'=(\log\frac{8}{9},0)\cap\{x:||x||_{\frac{1}{q}}<\frac{2}{q\cdot 2^k}\}$. We have the estimate

$$|G'|<card\{n:q\log\frac{8}{9}-\frac{2}{2^k}<n<\frac{2}{2^k}\}\cdot\frac{4}{q\cdot 2^k}\le(\frac{4}{2^k}+q\log\frac{9}{8})\cdot\frac{4}{q\cdot 2^k}.$$

Thus, for large values of k (that is, $k\ge K_0$), we have $|G'|<\frac{5}{2^k}$ and letting $H'=\{2^x: x\in G'\}$, we have $|H'|<\frac{5}{2^k}$ as well. We claim $I_F\cap\Phi(H_k)\subset H'$. To see this, suppose $y\in I_F\cap\Phi(H_k)$, that is, there exists n such that $ny\in H_k$. Set $x=\log y$. Since $\log\frac{8}{9}<x<0$ and $x+\log n\in G$, we have

$$||x+\log n||_{\frac{1}{q}}<\frac{1}{q\cdot 2^k},$$

and

$$\nu+\log\frac{8}{9}\le x+\log n<\nu.$$

Hence,

$$\nu-x+\log\frac{8}{9}\le\log n<\nu-x,$$

and using $0<-x<\log\frac{9}{8}$, we obtain

$$\nu+\log\frac{8}{9}<\log n<\nu+\log\frac{9}{8}.$$

Thus,

$$\frac{7}{8}2^{\nu}<\frac{8}{9}2^{\nu}<n<\frac{9}{8}2^{\nu}.$$

This implies $n=n_j$, for some j, $2^k<j\le L$ and therefore,

$$||-\log n||_{\frac{1}{q}}=||\log n||_{\frac{1}{q}}<\frac{1}{q\cdot 4\cdot 2^k};$$

so

$$||x||_{\frac{1}{q}}\le||x+\log n||_{\frac{1}{q}}+||-\log n||_{\frac{1}{q}}<\frac{1}{q\cdot 2^k}+\frac{1}{4\cdot q\cdot 2^k}<\frac{2}{q\cdot 2^k}.$$

We infer that $x\in G'$ and $y\in H'$. This completes the proof of the Lemma. ■

Questions

1. Is there a continous function f from the positive reals to the positive reals such that $|\{x : \sum_{n=1}^{\infty} f(nx) = +\infty\}| > 0$ and $|\{x : \sum_{n=1}^{\infty} f(nx) < +\infty\}| > 0$?

This first question relates back to the solutions of a problem of K. L. Chung [H-F].

2. Is there an unbounded countable subset G of the positive reals such that for every measurable map f of the positive reals into the nonnegative reals either for almost every $x, \sum_{g\in G} f(gx) = +\infty\}$ or else for almost every $x, \sum_{g\in G} f(gx) < \infty\}$

This second question is directly related to Haight's question in [H2].

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