On series of translates of positive functions II

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Abstract

In this paper we continue our investigation of series of the form $\sum_{\lambda\in\Lambda} f(x+\lambda)$. Given a sequence of natural numbers $n_1 < n_2 < \ldots$ we are interested in sets Λ of the form $\cup_{k=1}^{\infty}\alpha^k\mathbb{Z}\cap[n_k, n_{k+1})$, where $0<\alpha<1$. In case $\alpha = 1/q$, where $q>1$ is an integer, there is a zero-one law showing that for every measurable $f:\mathbb{R}\to\mathbb{R}^+$ the above sum either converges almost everywhere or diverges almost everywhere. However, for any other value of $\alpha\in(0,1)$ there is no such zero-one law.

1 Introduction

This paper is a continuation of papers [1], [2], and [3]. In [3], answering a question of Haight [6] and Weizsäcker [10] we showed that there is a measurable characteristic function $f:\mathbb{R}\to\mathbb{R}^+$ such that both of the sets $\{x:\sum_{n=1}^{\infty} f(nx) < +\infty\}$ and $\{x:\sum_{n=1}^{\infty} f(nx) = +\infty\}$ have positive measure. Earlier results, and further references related to convergence of $\sum f(nx)$ can be found in [7], [8] and [9]. In fact, Marstrand in [9] disproved the Khinchin conjecture [7] which concerned the convergence of Cesàro means $(1/k)\sum_{n=1}^{k} f(nx)$ to the integral of f on $[0,1]$, when f is the characteristic function of a set E with period 1.

In [1] and [2] we considered the following additive generalizations of the convergence problem of $\sum f(nx)$:

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Given Λ, an infinite discrete set of nonnegative numbers, and $f : \mathbb{R} \to \mathbb{R}$ a nonnegative Lebesgue measurable function we consider the sum of translates of f:

$$s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda) \tag{1}$$

and the sets of convergence and divergence $C(f, \lambda) = \{x : \sum_{\lambda\in\Lambda} f(x+\lambda) < +\infty\}$ and $D(f, \lambda) = \{x : \sum_{\lambda\in\Lambda} f(x + \lambda) = +\infty\}$.

A set Λ is of type 1 if either $C(f, \Lambda) = \mathbb{R}$ almost everywhere, or $C(f, \Lambda) = \emptyset$ a.e. (in this case a zero-one law holds for the convergence of (1)). Otherwise, Λ is of type 2. Using additive terminology, the result of [3] says that $\Lambda^{\log n} := \{\log n : n = 1, 2, ...\}$ is of type 2. In [1] and [2] several generalizations were considered. Given a sequence of natural numbers $n_1 < n_2 < ...$, it was shown that $\Lambda^{(\frac{1}{2})^k} := \cup_{k=1}^{\infty} \Lambda_k^{(\frac{1}{2})^k}$, where $\Lambda_k^{(\frac{1}{2})^k} := (\frac{1}{2})^k \mathbb{Z} \cap [n_k, n_{k+1})$ is of type 1. It was also shown that this type 1 set could be adjusted to become a type 2 set by making it become asymptotically dense at a faster rate as follows: there is a sequence $m(k) \to \infty$ such that $\Lambda^{(\frac{1}{2})^{m(k)}} := \cup_{k=1}^{\infty} \Lambda_k^{(\frac{1}{2})^{m(k)}}$, $\Lambda^{(\frac{1}{2})^{m(k)}} := (\frac{1}{2})^{m(k)} \mathbb{Z} \cap [n_k, n_{k+1})$ is of type 2.

A number $t > 0$ is called a translator of Λ if $(\Lambda + t) \setminus \Lambda$ is finite. Condition $(*)$ is said to be satisfied if $T(\Lambda)$, the countable additive semigroup of translators of Λ, is dense in $\mathbb{R}^+$. We showed that condition $(*)$ implies that $C(f, \Lambda)$ is either $\emptyset$, $\mathbb{R}$, or a closed left half-line modulo sets of measure zero. The two sets just described above both satisfy condition $(*)$. So condition $(*)$ is not enough to determine whether Λ is of type 1 or type 2.

In this paper for a given $\alpha \in (0, 1)$, we are interested in the sets $\Lambda^{\alpha^k} := \cup_{k=1}^{\infty} \Lambda_k^{\alpha^k}$, $\Lambda_k^{\alpha^k} = \alpha^k \mathbb{Z} \cap [n_k, n_{k+1})$.

If $\alpha = \frac{1}{q}$ for some $q \in \{2, 3, ...\}$, then a we indicate how a slight modification of the proof of Theorem 1 of [1] shows that $\Lambda^{(\frac{1}{q})^k}$ is of type 1 and condition $(*)$ is satisfied. If $\alpha \notin \mathbb{Q}$, then we indicate how one can apply Theorem 5 of [1] to show that Λ^{α^k} is of type 2.

Thus, there remains the difficult case when $\alpha = \frac{p}{q}$ with $(p, q) = 1$, $p, q > 1$, $p < q$. In this case we show that $\Lambda^{(\frac{p}{q})^k}$ is of type 2. The problem of showing this is not easy even in the special case when $p = 2$, $q = 3$. When working on this problem, for a while it seemed that we needed some information on the distribution of $\{(\frac{3}{2})^k\}$, (where $\{\cdot\}$ denotes the fractional part). To our surprize, and showing why the case $\Lambda^{(\frac{2}{3})^k}$ was difficult, it turned out, [4] that in 1980 it was not even known, whether $\{(\frac{3}{2})^k\}$ is uniformly distributed, or even dense in $[0, 1]$. These questions were extensively studied previously (see further references in [4]) and according to a recent letter from Choquet remain open at this time. Fortunately, we found a way avoiding any information about the distribution of $\{(\frac{3}{2})^k\}$, (or of $\{(\frac{q}{p})^k\}$) to determine that $\Lambda^{(\frac{2}{3})^k}$, (or $\Lambda^{(\frac{p}{q})^k}$) is of type 2. Another novelty of this paper is that in the cases $\Lambda^{(\frac{p}{q})^k}$, $(p > 1)$ condition $(*)$ is not satisfied and in Theorem 3 we also show that there exists a characteristic function f such that $C(f,\Lambda)$ does not equal $\emptyset$, $\mathbb{R}$, or a left half-line modulo sets of measure zero. This structure of $C(f,\Lambda)$ has not been seen before and casts a little more light on the question of what the possible structure could be.

Throughout the paper we assume that $n_1 < n_2 < ...$ is a given arbitrary monotone increasing sequence of natural numbers. We use the notation $\lfloor x \rfloor$ for the integer part of x and $|A|$ for the Lebesgue measure of the set $A \subset \mathbb{R}$.

2 The cases when $\alpha \notin \mathbb{Q}$, or $\alpha = 1/q$

Recall first Theorem 5 of [1]:

Theorem 1. *Suppose that there exist three intervals I, J, K such that $J = K + I - I$ (algebraic sum), I is to the left of J, and $dist(I,J) \geq |I|$, and two sequences (y_j) and (N_j) tending to infinity ($y_j \in \mathbb{R}^+$, $N_j \in \mathbb{N}$) such that, for each j, $y_j - I$ contains a set of N_j points of Λ independent from $\Lambda \cap (y_j - J)$ in the sense that the additive groups generated by these sets have only 0 in common. Then Λ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, $D(f,\Lambda)$ contains I and $C(f,\Lambda)$ has full measure on K.*

If $\alpha \notin \mathbb{Q}$ then one can apply Theorem 1 above. Indeed, choose intervals $I = [0, \frac{1}{16}]$, $K = [\frac{1}{2} - \frac{1}{16}, \frac{1}{2} + \frac{1}{16}]$, and $J = [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}]$. Set $y_j = (n_j + \frac{1}{8})$. Observe that $y_j - I \cap \Lambda^{\alpha^k} = \alpha^j \mathbb{Z} \cap (y_j - I)$, $y_j - J \cap \Lambda^{\alpha^k} = \alpha^{j-1}\mathbb{Z} \cap (y_j - J)$ and the additive groups $\alpha^j \mathbb{Z}$ and $\alpha^{j-1}\mathbb{Z}$ have only 0 in common. Setting $N_j = \#(y_j - I \cap \Lambda^{\alpha^k})$ we have $N_j \to \infty$ and hence all conditions of Theorem 1 are satisfied.

If $\alpha = 1/q$ one can repeat the proof of Theorem 1 in [1] by using the following modification of Lemma 2 of [1]. We leave the details to the reader.

Lemma 2. *Let $\varphi_n : \mathbb{T} \to \mathbb{R}$ be a sequence of positive measurable functions, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $\sum_{n=1}^{\infty} \varphi_n(q^n t) < \infty$ a.e., it then follows that $\sum_{n=1}^{\infty} \varphi_n(q^n t + \frac{k}{q}) < \infty$ for $k = 1, ..., q-1$.*

3 The case $\alpha = \frac{p}{q}$

¿From here on we work with fixed relative prime integers $p, q > 1$ and $1 < p < q$. For ease of notation we denote $\Lambda^{(\frac{p}{q})^k}$ by Λ and $\Lambda_k^{(\frac{p}{q})^k}$ by Λ_k.

Fix an integer r such that $r > 8$, $(r,p) = 1$, and $(r,q) = 1$.

The main result of our paper is the following:

Theorem 3. *The set Λ defined above is of type 2. Moreover, there exists a characteristic function $f : \mathbb{R} \to \mathbb{R}^+$ such that almost every point in $[\frac{1}{r}, \frac{2}{r}]$ belongs to $C(f,\Lambda)$, $|D(f,\Lambda) \cap [1 + \frac{1}{r}, 1 + \frac{2}{r}]| > \frac{1}{8r}$ and $|D(f,\Lambda) \cap [-1 + \frac{1}{r}, -1 + \frac{2}{r}]| > \frac{1}{8r}$.*

The proof of Theorem 3 relies on the following technical lemma.

Lemma 4. *Given positive integers M and K_{M-1} we can choose the sets $L^M \subset [\frac{1}{r}, \frac{2}{r}]$, $H^{1,M} \subset [1+\frac{1}{r}, 1+\frac{2}{r}]$, $H^{-1,M} \subset [-1+\frac{1}{r}, -1+\frac{2}{r}]$ and an integer $K_M > K_{M-1}$ such that*

$$|L^M| < \frac{2^{-M}}{r}, \tag{2}$$

$$|H^{1,M}| \geq \frac{1}{8r}, \quad |H^{-1,M}| \geq \frac{1}{8r} \tag{3}$$

and a characteristic function $f_M = \mathbb{1}_{E_M} : \mathbb{R} \to \mathbb{R}^+$ such that letting

$$\Lambda_{K_{M-1},K_M} = \cup_{k=K_{M-1}+1}^{K_M} \Lambda_k$$

and

$$s_{M-1,M}(x) = \sum_{\lambda \in \Lambda_{K_{M-1},K_M}} f_M(x+\lambda)$$

we have

$$s_{M-1,M}(x) = 0, \text{ for } x \in \left[\frac{1}{r}, \frac{2}{r}\right) \setminus L^M, \tag{4}$$

$$s_{M-1,M}(x) \geq 1, \text{ for } x \in H^{1,M} \cup H^{-1,M} \tag{5}$$

and $f_M = 0$ on $(-\infty, n_{K_{M-1}+2}) \cup (n_{K_M-2}, +\infty) \supset (-\infty, n_{K_{M-1}}+2) \cup (n_{K_M} - 2, +\infty)$.

First, let us prove Theorem 3 assuming Lemma 4 holds.

Proof of Theorem 3. Set $M = 1$ and $K_0 = 1$. Using induction and repeated application of Lemma 4, choose the sets L^M, $H^{1,M}$, $H^{-1,M}$, the functions f_M and the constants K_M for $M = 1, 2, \ldots$. Observe the sets E_M are pairwise disjoint. Set $f(x) = \sum_{M=1}^{\infty} f_M(x) = \mathbb{1}_{\cup_M E_M}(x)$. Also, observe that if $x \in [-1, 2]$, then $s(x) = \sum_{\lambda\in\Lambda_1} f(x+\lambda) + \sum_{M=1}^{\infty} s_{M-1,M}(x)$ on $[-1, 2]$ and hence by the Borel-Cantelli lemma, (2) and (4) imply that $s(x) < \infty$ for almost every point of $[\frac{1}{r}, \frac{2}{r}]$. On the other hand, (5) implies that $s(x) = \infty$ for $\cap_{N=1}^{\infty} \cup_{M=N}^{\infty} H^{1,M} \cup H^{-1,M}$. Using (3), we have

$$\left| \bigcap_{N=1}^{\infty} \bigcup_{M=N}^{\infty} H^{1,M} \cap \left[1+\frac{1}{r}, 1+\frac{2}{r}\right] \right| \geq \frac{1}{8r},$$

and

$$\left| \bigcap_{N=1}^{\infty} \bigcup_{M=N}^{\infty} H^{-1,M} \cap \left[-1+\frac{1}{r}, -1+\frac{2}{r}\right] \right| \geq \frac{1}{8r}.$$

□

PROOF OF LEMMA 4. Set $m = 2^{M+2}$ and choose $k_1 > K_{M-1} + 2$ such that $(\frac{p}{q})^{k_1 - 1} < \frac{1}{8r}$. Assuming k_{j-1} has been defined choose $k_j > k_{j-1}$ such that

$$q^{k_j - k_{j-1}} > \frac{10p}{(\frac{p}{q})^{k_j}}. \tag{6}$$

Do this for $j = 2, ..., m$ and put $K_M = k_m + 4$.

Set

$$N_j = \left\lfloor \frac{3}{(\frac{p}{q})^{k_j}} \right\rfloor \tag{7}$$

and assume $\tau \in [0, \frac{1}{r})$ is fixed. Let

$$\Phi(\tau) = \{\tau + l_1(\frac{p}{q})^{k_1} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t : 1 \leq l_j \leq N_j, t \in \mathbb{Z}\}. \tag{8}$$

Note that $\Phi(\tau)$ has period $\frac{1}{r}$ and next we show that its elements have a certain "independence" property.

Assume $x \in \Phi(\tau)$ has the representations:

$$x = \tau + l_1(\frac{p}{q})^{k_1} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t = \tau + l'_1(\frac{p}{q})^{k_1} + ... + l'_m(\frac{p}{q})^{k_m} + \frac{1}{r}t'. \tag{9}$$

Then, after multiplying by $r \cdot q^{k_m}$ and rearranging we obtain $q^{k_m}(t - t') = A_1 r$, where A_1 is an integer. Since $(q, r) = 1$ we have $r | t - t'$ and hence $\frac{1}{r}t - \frac{1}{r}t' \in \mathbb{Z}$.

Now, assume $j \leq m$ and we have already verified that $l_i = l'_i$ for $i > j$. Then, multiplying (9) by q^{k_j} and rearranging we obtain that $(l_j - l'_j)p^{k_j} = A_2 \cdot q^{k_j - k_{j-1}}$, where $A_2 \in \mathbb{Z}$. Hence, $q^{k_j - k_{j-1}} | l_j - l'_j$, but by (6), (7), and (8) we have $|l_j - l'_j| < q^{k_j - k_{j-1}}$. Therefore, $l_j = l'_j$. Thus each x in $\Phi(\tau)$ has a unique representation given by (9). This implies

$$\# \left(\Phi(\tau) \cap \left[\alpha, \alpha + \frac{1}{r}\right) \right) = N_1 \cdots N_m \text{ for arbitrary } \alpha \in \mathbb{R}. \tag{10}$$

The set $\Phi(\tau)$ will be used to define those points in $[\frac{1}{r}, \frac{2}{r})$ where $s_{M-1,M}(x)$ can take values different from 0. In the sequel it will be enlarged to obtain a set of positive measure. Set $S_j = \lfloor (\frac{p}{q})^{-k_j} \rfloor + 1$.

For even j choose the integer ν_{k_j} such that $n_{k_j} \leq \nu_{k_j}(\frac{p}{q})^{k_j} < n_{k_j} + (\frac{p}{q})^{k_j}$ and set $J_{k_j} = [n_{k_j} + \frac{1}{2}, n_{k_j} + 1)$. For odd j choose the integer ν_{k_j} such that $n_{k_j+1} - 1 \leq \nu_{k_j}(\frac{p}{q})^{k_j} < n_{k_j+1} - 1 + (\frac{p}{q})^{k_j}$ and set $J_{k_j} = [n_{k_j+1} - \frac{1}{2}, n_{k_j+1})$.

Observe that if $\lambda \in \Lambda_{k_j} \cap J_{k_j}$, then λ has the form $(\nu_{k_j} + l)(\frac{p}{q})^{k_j}$, where $l \in \{0, ..., S_j\}$. To see this, note that we have $\lambda = (\nu_{k_j} + l)(\frac{p}{q})^{k_j} < n_{k_j} + 1$. So, $l < 1 + 1/(p/q)^{k_j}$. Or, $l - 1 < 1/(p/q)^{k_j}$. Since $1/(p/q)^{k_j}$ is not an integer, $l - 1 \leq \lfloor 1/(p/q)^{k_j} \rfloor$. Thus, $l \leq S_j$.

Put

$$\Theta_j(\tau) = \{\tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j} + l_j)(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t : \qquad (11)$$
$$l_i \in \{1, ..., N_i\} \text{ for } i \neq j,\ l_j \in \{S_j + 1, ..., N_j\},\ t \in \mathbb{Z}\}.$$

In the sequel we will choose certain values of τ and fatten up $\Theta_j(\tau) \cap J_{k_j}$ to be the collection of those points where $f_M = 1$ in $[n_{k_j}, n_{k_j+1})$.

Note that if $x \in (0, \frac{1}{2})$ and $x + \lambda \in J_{k_j}$ for some $\lambda \in \Lambda$, then

$$\lambda = (\nu_{k_j} + \bar{l}_j)(\frac{p}{q})^{k_j} \text{ for some } \bar{l}_j \in \{0, ..., S_j\}. \qquad (12)$$

Thus, if $x + \lambda \in \Theta_j(\tau) \cap J_{k_j}$ (that is, $f_M(x) = 1$), then we would have

$$x + (\nu_{k_j} + \bar{l}_j)(\frac{p}{q})^{k_j} = \tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j} + l_j)(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t;$$

and therefore,

$$x = \tau + l_1(\frac{p}{q})^{k_1} + ... + (l_j - \bar{l}_j)(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t.$$

By (11) and (12), $l_j - \bar{l}_j \in \{1, ..., N_j\}$ and hence

$$x \in (0, \frac{1}{2}) \text{ and } x + \lambda \in \Theta_j(\tau) \cap J_{k_j} \text{ imply } x \in \Phi(\tau). \qquad (13)$$

Next we fix the values of the integers $\bar{\bar{l}}_{j,1}$ and $\bar{\bar{l}}_{j,-1}$ so that

$$\text{i) } p \not| \bar{\bar{l}}_{j,1} \text{ and } q \not| \bar{\bar{l}}_{j,-1}, \qquad (14)$$
$$\text{ii) } \lambda_{j,1} := \bar{\bar{l}}_{j,1}\left(\frac{p}{q}\right)^{k_j-1} \in \Lambda_{k_j-1} \cap [n_{k_j} - 1, n_{k_j}),$$
$$\lambda_{j,-1} := \bar{\bar{l}}_{j,-1}\left(\frac{p}{q}\right)^{k_j+1} \in \Lambda_{k_j+1} \cap [n_{k_j+1}, n_{k_j+1} + 1),$$
$$\text{iii) } \left[1 + \frac{1}{r}, 1 + \frac{2}{r}\right] + \lambda_{j,1} \subset \left[n_{k_j} + 1 - \frac{4}{r}, n_{k_j} + 1 - \frac{2}{r}\right],$$
$$\text{iv) } \left[-1 + \frac{1}{r}, -1 + \frac{2}{r}\right] + \lambda_{j,-1} \subset \left[n_{k_j+1} - \frac{4}{r}, n_{k_j+1} - \frac{2}{r}\right].$$

Observe that by iii) $[1 + \frac{1}{r}, 1 + \frac{2}{r}] + \lambda_{j,1} \subset J_{k_j}$ when j is even, and by iv) $[-1 + \frac{1}{r}, -1 + \frac{2}{r}] + \lambda_{j,-1} \subset J_{k_j}$ when j is odd.

Set

$$\Psi_{j,1}(\tau) = ((\Theta_j(\tau) \cap J_{k_j}) - \lambda_{j,1}) \cap [1 + \frac{1}{r}, 1 + \frac{2}{r}), \qquad (15)$$

and

$$\Psi_{j,-1}(\tau) = ((\Theta_j(\tau) \cap J_{k_j}) - \lambda_{j,-1}) \cap [-1 + \frac{1}{r}, -1 + \frac{2}{r}). \tag{16}$$

These sets will be enlarged by adding a small interval to define $H^{1,M}$ and $H^{-1,M}$ and will give points $x \in [-1 + \frac{1}{r}, -1 + \frac{2}{r}] \cup [1 + \frac{1}{r}, 1 + \frac{2}{r})$ for which $f_M(x + \lambda_{j,-1}) = 1$ or $f_M(x + \lambda_{j,1}) = 1$, depending on the parity of j.

Arguing as at (10), using (11) and (7) one can show that

$$\#\left(\Theta_j(\tau) \cap \left[\alpha, \alpha + \frac{1}{r}\right)\right) > \frac{N_1 \cdots N_m}{2}$$

holds for all $\alpha \in \mathbb{R}$. This implies for even j and for all $\alpha \in \mathbb{R}$

$$\#\Psi_{j,1}(\tau) > \frac{N_1 \cdots N_m}{2} = \frac{\#(\Phi(\tau) \cap [\alpha, \alpha + \frac{1}{r}))}{2}, \tag{17}$$

for odd j and for all $\alpha \in \mathbb{R}$

$$\#\Psi_{j,-1}(\tau) > \frac{N_1 \cdots N_m}{2} = \frac{\#(\Phi(\tau) \cap [\alpha, \alpha + \frac{1}{r}))}{2}. \tag{18}$$

Next we want to show that if $j' < j$, j' and j are both even then

$$\Psi_{j,1}(\tau) \cap \Psi_{j',1}(\tau) = \emptyset, \tag{19}$$

and if $j' < j$, j' and j are both odd then

$$\Psi_{j,-1}(\tau) \cap \Psi_{j',-1}(\tau) = \emptyset. \tag{20}$$

This will be the "independence" property of the sets $\Psi_{j,\pm 1}$. Since they are disjoint we will obtain more than $(m/2) \cdot N_1 \cdots N_m/2 = (m/4)\#(\Phi(\tau) \cap [\frac{1}{r}, \frac{2}{r}))$ points in $[-1 + \frac{1}{r}, -1 + \frac{2}{r})$, and in $[1 + \frac{1}{r}, 1 + \frac{2}{r})$, where $\sum_{\lambda \in \Lambda_{M-1,M}} f(x + \lambda) \geq 1$ will hold.

First we verify (19). Proceeding towards a contradiction assume $x \in \Psi_{j,1}(\tau) \cap \Psi_{j',1}(\tau)$. Then there exist

$$t, t' \in \mathbb{Z},\ l_i \in \{1, ..., N_i\} \text{ for } i \neq j, l'_i \in \{1, ..., N_i\} \text{ for } i \neq j', \tag{21}$$

$$l_j \in \{S_j + 1, ..., N_j\} \text{ and } \ l'_{j'} \in \{S_{j'} + 1, ..., N_{j'}\} \text{ such that} \tag{22}$$

$$\tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j} + l_j - \bar{\bar{l}}_{j,1}\frac{q}{p})(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t = \tag{23}$$

$$\tau + l'_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_{j'}} + l'_{j'} - \bar{\bar{l}}_{j',1}\frac{q}{p})(\frac{p}{q})^{k_{j'}} + ... + l'_m(\frac{p}{q})^{k_m} + \frac{1}{r}t'.$$

Again, as we argued earlier, after multiplying by $r \cdot q^{k_m}$ and rearranging we obtain $q^{k_m}(t - t') = A_3 \cdot r$ where $A_3 \in \mathbb{Z}$. Using $(q, r) = 1$ we have $r | t - t'$. Therefore, $\frac{1}{r}t - \frac{1}{r}t' \in \mathbb{Z}$.

Now assume $i \in \{j+1, ..., m\}$ and we have shown $l_\iota = l'_\iota$ for $\iota > i$. Multiplying (23) by q^{k_i} and rearranging we infer $(l_i - l'_i)p^{k_i} = A_4 q^{k_i - k_{i-1}}$ where $A_4 \in \mathbb{Z}$.

Since $(p,q) = 1$ we again find that $q^{k_i - k_{i-1}} | l_i - l'_i$. But by (21) and (6) we have $|l_i - l'_i| < q^{k_i - k_{i-1}}$, hence $l_i = l'_i$.

Now we turn to the case when $i = j$ and recall that in this case j is even. This time after multiplying (23) by q^{k_j} and rearranging we have

$$(\nu_{k_j} + l_j - \bar{\bar{l}}_{j,1}\frac{q}{p} - l'_j)p^{k_j} = q^{k_j - k_{j-1}} A_5 \tag{24}$$

where $A_5 \in \mathbb{Z}$.

Hence,

$$q^{k_j - k_{j-1}} | (p(\nu_{k_j} + l_j - l'_j) - q\bar{\bar{l}}_{j,1})p^{k_j - 1}. \tag{25}$$

Since $(p,q) = 1$, we obtain

$$q^{k_j - k_{j-1}} | p(\nu_{k_j} + l_j - l'_j) - q\bar{\bar{l}}_{j,1} := B_1. \tag{26}$$

Recall that

$$n_{k_j} \leq \nu_{k_j}\left(\frac{p}{q}\right)^{k_j} < n_{k_j} + (\frac{p}{q})^{k_j} \text{ and } n_{k_j} - 1 \leq \bar{\bar{l}}_{j,1}\left(\frac{p}{q}\right)^{k_j - 1} < n_{k_j}.$$

Thus,

$$0 < \nu_{k_j}\left(\frac{p}{q}\right)^{k_j} - \bar{\bar{l}}_{j,1}\left(\frac{p}{q}\right)^{k_j - 1} < 2$$

and

$$0 < p\nu_{k_j} - \bar{\bar{l}}_{j,1} q < \frac{2 \cdot q^{k_j}}{p^{k_j - 1}} = \frac{2p}{(\frac{p}{q})^{k_j}}.$$

Using (22), (7) and (6) we infer $|B_1| < q^{k_j - k_{j-1}}$ and (26) implies $B_1 = 0$, but this is impossible since by (14) i), we assumed $p \not| \bar{\bar{l}}_{j,1}$. This proves (19).

To verify (20) we use an argument which is simpler and similar to the case (19). We just outline the differences of the arguments. So, (23) is replaced by

$$\begin{aligned} &\tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j} + l_j - \bar{\bar{l}}_{j,-1}\frac{p}{q})(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t = \\ &\tau + l'_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_{j'}} + l'_{j'} - \bar{\bar{l}}_{j',-1}\frac{p}{q})(\frac{p}{q})^{k_{j'}} + ... + l'_m(\frac{p}{q})^{k_m} + \frac{1}{r}t'. \end{aligned} \tag{27}$$

One can show again that $\frac{1}{r}t - \frac{1}{r}t' \in \mathbb{Z}$ and for $i \in \{j+1, ..., m\}$, $l_i = l'_i$. When $i = j$ we need to replace (25) by

$$q^{k_j - k_{j-1}} | (\nu_{k_j} + l_j - l'_j)p^{k_j} - \bar{\bar{l}}_{j,-1}\frac{p^{k_j + 1}}{q} \tag{28}$$

which contradicts (14) i), since by this assumption the right hand side of the above formula is not an integer.

Now we introduce
$\widetilde{\Psi}_{j,1}(\tau)=\Psi_{j,1}(\tau)+\frac{1}{r}t,\ t\in\mathbb{Z}$ and $\widetilde{\Psi}_{j,-1}(\tau)=\Psi_{j,1}(\tau)+\frac{1}{r}t,\ t\in\mathbb{Z}$, which are the $\frac{1}{r}$ periodic extensions of $\Psi_{j,1}(\tau)$ and $\Psi_{j,-1}(\tau)$.

Our aim is to show that if $\epsilon>0$ is sufficiently small and we add an interval I_ϵ of length ϵ to these discrete periodic sets then we have the required function. For $\epsilon>0$ set $I_\epsilon=[0,\epsilon]$ and put
$\Phi'(\tau)=(\Phi(\tau)+I_\epsilon)\cap[\frac{1}{r},\frac{2}{r})$,
$\Psi'_1(\tau)=(\cup_{j=1}^m(\widetilde{\Psi}_{j,1}(\tau)+I_\epsilon))\cap[1+\frac{1}{r},1+\frac{2}{r})$,
$\Psi'_{-1}(\tau)=(\cup_{j=1}^m(\widetilde{\Psi}_{j,-1}(\tau)+I_\epsilon))\cap[-1+\frac{1}{r},-1+\frac{2}{r})$.
Now, next we want to take advantage of the estimates (10), (17), and (18) to obtain measure estimate (29) below. In our notation usually we denote by $'$ those sets which are obtained from discrete point sets after the addition of I_ϵ.

Indeed, using (10), and (17-20) one can easily see that if $\epsilon>0$ is chosen sufficiently small then we have

$$|\Psi'_1(\tau)|>\frac{m}{4}|\Phi'(\tau)| \text{ and } |\Psi'_{-1}(\tau)|>\frac{m}{4}|\Phi'(\tau)|. \tag{29}$$

We now fix such a small $\epsilon>0$ and set
$\Theta'(\tau)=\cup_{j=1}^m(\Theta_j(\tau)+I_\epsilon)\cap J_{k_j}$.

Observe that from (13) it follows that

$$\text{if } x\in\left[\frac{1}{r},\frac{2}{r}\right),\ \text{and } 1\!\!1_{\Theta'(\tau)}(x+\lambda)=1 \text{ then } x\in\Phi'(\tau). \tag{30}$$

By (15) and (16) it is also clear that from $x\in\Psi'_1(\tau)$, or from $x\in\Psi'_{-1}(\tau)$, it follows that

$$\sum_{j=1}^m 1\!\!1_{\Theta'(\tau)}(x+\lambda_{j,1})\geq 1, \text{ or } \sum_{j=1}^m 1\!\!1_{\Theta'(\tau)}(x+\lambda_{j,-1})\geq 1.$$

Hence

$$\sum_{\lambda\in\Lambda_{K_{M-1},K_M}} 1\!\!1_{\Theta'(\tau)}(x+\lambda)\geq 1 \tag{31}$$

for these $x\in\Psi'_1(\tau)\cup\Psi'_{-1}(\tau)$.

Next we choose translations $0\leq\tau_{i,1},\ \tau_{i,-1}<\frac{1}{r},\ i=1,...,\gamma$, where the number γ will be determined later. Set $\tau_{1,1}=\tau_{1,-1}=0$. Assume that we have already defined $\tau_{i,1}$, and $\tau_{i,-1}$ for $i=1,...,j$ for some $j\geq 1$ so that if
$H'_{j,1}=\cup_{i=1}^j\Psi'_1(\tau_{i,1})$, and $H'_{j,-1}=\cup_{i=1}^j\Psi'_{-1}(\tau_{i,-1})$, then

$$\begin{aligned}&\text{from } |H'_{j,1}|<\frac{1}{8}\frac{1}{r} \text{ it follows that } |H'_{j,1}|\geq\frac{jm}{8}|\Phi'(0)|, \text{ and}\\ &\text{from } |H'_{j,-1}|<\frac{1}{8}\frac{1}{r} \text{ it follows that } |H'_{j,-1}|\geq\frac{jm}{8}|\Phi'(0)|.\end{aligned} \tag{32}$$

By choosing $\epsilon>0$ sufficiently small we can assume $|H'_{1,\pm 1}|<\frac{1}{8}\frac{1}{r}$.

We also put $L'_{j,1} = \cup_{i=1}^{j} \Phi'(\tau_{i,1})$, $L'_{j,-1} = \cup_{i=1}^{j} \Phi'(\tau_{i,-1})$.

Notice that $|\Phi'(\tau)| = |\Phi'(0)|$, $|\Psi'_1(\tau)| = |\Psi'_1(0)|$, and $|\Psi'_{-1}(\tau)| = |\Psi'_{-1}(0)|$ for all $\tau \in [0, \frac{1}{r})$. It is clear that (32) holds when $j = 1$.

The definitions of $L'_{j,1}$ and $L'_{j,-1}$ imply that $|L'_{j,1}| \leq j|\Phi'(0)|$, and $|L'_{j,-1}| \leq j|\Phi'(0)|$.

If $|H'_{j,1}| < \frac{1}{8}\frac{1}{r}$ then $\frac{1}{8}\frac{1}{r} > |H'_{j,1}| \geq \frac{jm}{8}|\Phi'(0)| \geq \frac{m}{8}|L'_{j,1}|$. Similarly, if $|H'_{j,-1}| < \frac{1}{8}\frac{1}{r}$ then $\frac{1}{8}\frac{1}{r} > |H'_{j,-1}| \geq \frac{jm}{8}|\Phi'(0)| \geq \frac{m}{8}|L'_{j,-1}|$. This implies

$$2^{-M-2} \cdot \frac{1}{r} > |L'_{j,1}|, \text{ and } 2^{-M-2} \cdot \frac{1}{r} > |L'_{j,-1}|. \tag{33}$$

Next we show how to choose $\tau_{j+1,1}$ and $\tau_{j+1,-1}$.

Assume $|H'_{j,1}| < \frac{1}{8}\frac{1}{r}$. We claim that there exists $\tau_{j+1,1} \in [0, \frac{1}{r})$ such that

$$|H'_{j,1} \cap \Psi'_1(\tau_{j+1,1})| < \frac{1}{2}|\Psi'_1(\tau_{j+1,1})|. \tag{34}$$

In the sequel $\sigma +_r \tau$ denotes the sum of σ and τ modulo $\frac{1}{r}$. We have

$$r\int_0^{\frac{1}{r}} |H'_{j,1} \cap \Psi'_1(\tau)|d\tau = r\int_0^{\frac{1}{r}} \int_0^{\frac{1}{r}} \mathbb{1}_{H'_{j,1}}(\sigma) \cdot \mathbb{1}_{\Psi'_1(0)+_r\tau}(\sigma)d\sigma d\tau =$$

$$r\int_0^{\frac{1}{r}} \int_0^{\frac{1}{r}} \mathbb{1}_{H'_{j,1}}(\sigma) \cdot \mathbb{1}_{\Psi'_1(0)}(\sigma +_r (-\tau))d\sigma d\tau =$$

$$r\int_0^{\frac{1}{r}} \int_0^{\frac{1}{r}} \mathbb{1}_{H'_{j,1}}(\sigma' +_r \tau) \cdot \mathbb{1}_{\Psi'_1(0)}(\sigma')d\sigma' d\tau = r|H'_{j,1}| \cdot |\Psi'_1(0)| < \frac{1}{8}|\Psi'_1(0)|.$$

This implies that there exists $\tau_{j+1,1}$ for which (34) holds.

Now,

$$|H'_{j+1,1}| = |H'_{j,1} \cup (\Psi'_1(\tau_{j+1,1}) \setminus H'_{j,1})| >$$

$$\frac{jm}{8}|\Phi'(0)| + \frac{1}{2}|\Psi'_1(\tau_{j+1,1})| >$$

$$\frac{jm}{8}|\Phi'(0)| + \frac{1}{2}\frac{m}{4}|\Phi'(0)| = (j+1)\frac{m}{8}|\Phi'(0)|,$$

where we also used (29).

This will yield (32) with j replaced by $j+1$ when $|H'_{j+1,1}| < \frac{1}{8}\frac{1}{r}$. Also, observe that if $|H'_{j+1,1}| \geq \frac{1}{8}\frac{1}{r}$ then, recalling (33),
$|\Phi'(\tau_{j+1,1})| = |\Phi'(\tau_{j,1})| \leq |L'_{j,1}| < 2^{-M-2} \cdot \frac{1}{r}$ implies

$$|L'_{j+1,1}| = |L'_{j,1} \cup \Phi'(\tau_{j+1,1})| < 2^{-M-1} \cdot \frac{1}{r}. \tag{35}$$

¿From (32) it follows that we reach a step j when $|H'_{j+1,1}| \geq \frac{1}{8}\frac{1}{r}$ but $|H'_{j,1}| < \frac{1}{8}\frac{1}{r}$. For this value of j set $\gamma_1 = j+1$. By (35) we have $|L'_{\gamma_1,1}| < 2^{-M-1} \cdot \frac{1}{r}$.

Argue similarly for choosing $\tau_{j+1,-1}$ and choose γ_{-1} as γ_1 was chosen.

Set $\gamma = \max(\gamma_1, \gamma_{-1})$. If $\gamma > \gamma_{-1}$ then for $\gamma_{-1} < j \leq \gamma$ set $\tau_{j,-1} = \tau_{1,-1} = 0$. This implies that $L'_{j,-1} = L'_{\gamma_{-1},-1}$ for $\gamma_{-1} \leq j \leq \gamma$. Similarly if $\gamma > \gamma_1$ then for $\gamma_1 < j \leq \gamma$ set $\tau_{j,1} = \tau_{1,1} = 0$. This implies that $L'_{j,1} = L'_{\gamma_1,1}$ for $\gamma_1 \leq j \leq \gamma$.

Set $L^M = L'_{\gamma,1} \cup L'_{\gamma,-1}$. We have
$|L^M| \leq |L'_{\gamma_1,1}| + |L'_{\gamma_{-1},-1}| < 2^{-M} \cdot \frac{1}{r}$.

Put $H^{1,M} = H'_{\gamma,1} = H'_{\gamma_1,1}$, and $H^{-1,M} = H'_{\gamma,-1} = H'_{\gamma_{-1},-1}$. Hence $|H^{1,M}| \geq \frac{1}{8} \cdot \frac{1}{r}$, $|H^{-1,M}| \geq \frac{1}{8} \cdot \frac{1}{r}$.

Finally, set $f_M = \sum_{j=1}^{\gamma}(\mathbb{1}_{\Theta'(\tau_{j,1})} + \mathbb{1}_{\Theta'(\tau_{j,-1})})$. Now (30) implies that if $x \in [\frac{1}{r}, \frac{2}{r})$ and $f_M(x+\lambda) = 1$ then $x \in \cup_{j=1}^{\gamma}(\Phi'(\tau_{j,1}) \cup \Phi'(\tau_{j,-1})) = L'_{\gamma,1} \cup L'_{\gamma,-1} = L^M$. Hence $s_{M-1,M}(x) = 0$ for $x \in [\frac{1}{r}, \frac{2}{r}) \setminus L^M$.

By using (31) we find that if
$x \in \cup_{j=1}^{\gamma}(\Psi'_1(\tau_{j,1}) \cup \Psi'_{-1}(\tau_{j,-1})) = H^{1,M} \cup H^{-1,M}$, then $s_{M-1,M}(x) \geq 1$. This completes the proof of the lemma. □

References

[1] Z. Buczolich, J-P. Kahane and R.D. Mauldin, On series of translates of positive functions, *Acta Mathematica Hungarica*, (to appear).

[2] Z. Buczolich, J-P. Kahane and R.D. Mauldin, Sur les séries de fonctions positives, *C. R. Acad. Sci. Paris Sér. I. Math.* **329** (1999), no. 4, 261-264.

[3] Z. Buczolich and R.D. Mauldin, On the convergence of $\sum f(nx)$ for measurable functions, *Mathematika,* (to appear).

[4] Choquet, G., Construction effective de suites $(k(3/2)^n)$. Études des mesures (3/2)-stables, *C. R. Acad Sci. Paris Sér. A-B* **291** (1980), no. 2, A69-A74.

[5] J.A. Haight, A linear set of infinite measure with no two points having integral ratio, *Mathematika* **17** (1970), 133-138.

[6] J.A. Haight, A set of infinite measure whose ratio set does not contain a given sequence, *Mathematika* **22** (1975), 195-201.

[7] A. Khinchin, Ein Satz über Kettenbruche mit arithmetischen Anwendungen, *Math. Zeit.,* **18** (1923) 289-306.

[8] C. G. Lekkerkerker, Lattice points in unbounded point sets, I. *Indag. Math.,* **20** (1958) 197-205.

[9] J. M. Marstrand, On Khinchin's conjecture about strong uniform distribution, *Proc. London Math. Soc.* **3** 21 (1970) 540-56.

[10] H. v. Weizsäcker, *Zum Konvergenzverhalten der Reihe* $\Sigma_{n=1}^{\infty} f(nt)$ *für* λ-*messbare Funktionen* $f : \mathbb{R}^+ \to \mathbb{R}^+$, Diplomarbeit, Universität München, 1970.