to be explored, but the heavy technical details make progress difficult. Though mortally ill, Levinson himself energetically investigated some of the further ramifications of his work. Certainly his work will demand closer scrutiny in the years to come.

References

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2. ______, A simplification of the proof that $N_0(T)>(1/3)N(T)$ for Riemann's zeta-function, Advances in Math., 18 (1975) 239–242 (MR 53, #7966).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109.

MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

THE EXISTENCE OF NON-MEASURABLE SETS

R. DANIEL MAULDIN

Let X be a set and τ a locally compact T_2 topology of subsets of X. Let I be a nonnegative linear functional on $C_{00}(X)$ and let ι be the outer measure on X induced by I. These notations are taken from Hewitt and Stromberg [**1**]. In their book, on pages 134 and 135, it is stated that no general facts concerning the existence of non-ι-measurable sets or methods of constructing such sets are known. In this note, we hope to clear up this situation using only techniques which fall within the scope of those developed in [**1**].

In order to give the general solution, we will use the construction given in the following lemma.

LEMMA. *Let F be a compact subset of X such that $\iota(F)=k>0$ and $\iota(\{x\})=0$, for each x in F. Then there is a compact subset H of F such that $\iota(H)\geqslant k/2$ and a continuous map f of H onto the Cantor set, C, such that $\iota(f^{-1}(t))=0$, for each t in C.*

Proof. Let $H(0)$ and $H(1)$ be disjoint closed subsets of F such that $k/4<\iota(H(t))<k/2$, for $t=0$ or 1. An induction argument reveals that there is a system, $\{H(t_1,\dots,t_n)$: $(t_1,\dots,t_n)$ is a finite sequence of zeroes and ones$\}$, such that for each n, the sets $H(t_1,\dots,t_n)$ are pairwise disjoint closed sets with $k/2^{n+1}<\iota(H(t_1,\dots,t_n))<k/2^n$ and such that $H(t_1,\dots,t_n)$ contains $H(t_1,\dots,t_n,t)$, for $t=0$ or 1.

Now, let $H(n)=\cup\{H(t_1,\ldots,t_n): (t_1,\ldots,t_n)\in\{0,1\}^n\}$. Notice that each set $H(n)$ is a compact subset of F, $H(n)\supseteq H(n+1)$ and $\iota(H(n))>k/2$. Also notice that for each x in $H=\cap H(n)$, there is exactly one sequence $t=(t_1,t_2,t_3,\ldots)$ in $\{0,1\}^N$ such that $x\in\cap\{H(t_1,\ldots,t_n): n\in N\}$. So, define $f: H\to\{0,1\}^N$, by $f(x)=t$. It can be easily checked that f is onto $C=\{0,1\}^N$. (We identify the Cantor set with $\{0,1\}^N$ under the product topology.) It is easy to see that f is a continuous map and if $t=(t_1,t_2,t_3,\ldots)$ is a point of the Cantor set, then $f^{-1}(t)=\cap\{H(t_1,\ldots,t_n): n\in N\}$, which means that $\iota(f^{-1}(t))=0$. Finally, $\iota(H)\geqslant k/2$.

We can now state some conditions under which every subset of X is measurable.

THEOREM. *The following two statements are equivalent:*

(1) *every subset of X is ι-measurable; and* (2) *there is a subset D of X such that $X-D$ is a locally ι-null set and such that $D\cap F$ is countable, for each compact subset F of X.*

Proof. First, let us assume (1) holds. Let $D=\{x:\iota\{x\})>0\}$. Since ι is locally bounded (Theorem 9.5 of [**1**]), $D\cap F$ is countable, for each compact set F.

Suppose that $X-D$ is not locally null. Then there is a compact set T such that $\iota(T\cap(X-D))>0$. Since every subset of X is measurable, there is a compact set F contained in $T\cap(X-D)$ which has positive measure. Let H be a compact subset of F with positive measure and f a continuous map of H onto the Cantor set, C, as constructed in the lemma. Let W be a subset of C such that W and $C-W$ intersect every closed uncountable subset of C (Exercise 10.54 of [**1**]). Let $E_1=f^{-1}(W)$ and $E_2=f^{-1}(C-W)$. If H_1 is a compact set contained in E_1, then $f(H_1)$ is a compact set contained in W. Thus, $f(H_1)$ is countable. Since $f^{-1}(f(H_1))$ has measure zero, H_1 has measure zero. Since E_1 is measurable, $\iota(E_1)=0$. Similarly, $\iota(E_2)=0$. Therefore, $\iota(H)=0$. This contradiction establishes the fact that (1) implies (2).

Now, let us assume that (2) holds. If A is a subset of X, then $A\cap(X-D)$ is locally null and therefore $A\cap(X-D)$ is measurable (Corollary 10.32 of [**1**]). Also, $A\cap D$ is measurable, since $(A\cap D)\cap F$ is measurable for every compact set F (Theorem 10.31 of [**1**]). Therefore, A is measurable. Thus, (2) implies (1). Q.E.D.

COROLLARY. *If $\iota(\{x\})=0$, for each x in X and ι is not the zero measure, then there are nonmeasurable sets.*

Let us remark that if E is a subset of X such that $X-E$ is locally null, then $X-\bar{E}$ is of measure zero, and if X is σ-compact, then obviously $X-E$ is of measure zero.

The following example shows that it is possible that (2) holds and $\iota(X-D)=\infty$.

Example: Let $Y=\{0\}\cup\{1/n:n\in N\}$ and let Y have the relative topology as a subset of R, the reals. Then Y is a compact T_2 space. Define $\mu(A)=\Sigma\{1/n^2:1/n\in A\}$, for every subset A of Y. Let $X=R\times Y$, where R is the reals provided with the discrete topology and X has the product topology. Then X is a locally compact T_2 space. If f is in $C_{00}(X)$, then there are only finitely many t in R so that f_t is not the zero function on Y, where $f_t(y)=f(t,y)$. Define I on $C_{00}(X)$ by $I(f)=\Sigma\{\int f_t\,d\mu:t\in R\}$. It can be easily seen that I is a nonnegative linear functional on X and that every subset of X is measurable with respect to the induced measure ι. The set D of statement (2) of the theorem is $\{(t,y): y\neq 0\}$. Finally, it can be checked that $\iota(X-D)=\infty$.

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Reference

1. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer, New York, 1965.

DEPARTMENT OF MATHEMATICS, NORTH TEXAS STATE UNIVERSITY, DENTON, TX 76203.