REALIZATIONS OF MAPS

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1. INTRODUCTION

Let $(X,\mathscr{M},\lambda)$ and $(Y,\mathscr{A},\mu)$ be σ-finite measure spaces. For the results of this paper we can reduce our considerations to the case in which λ and μ are finite, and we assume that this reduction has been made. Throughout Sections 1–3 (but not in Section 4) we assume that μ is complete.

Let $\mathscr{E}$ be the measure algebra of $(Y,\mathscr{A},\mu)$, provided with the metric ρ given by $\rho([A],[B]) = \mu(A\triangle B)$, where $[A]$ is the measure class of A. We consider three types of maps $\phi: X \to \mathscr{E}$. First, we say that ϕ is "realizable" if there exists a measurable subset E of $X \times Y$ such that, for all $x \in X$, the section

$$E_x = \{y \in Y : (x,y) \in E\}$$

is in the measure class $\phi(x)$; such a set E will be called a "realization" of ϕ. Here the measure on $X \times Y$ is the completed product measure $\lambda \times \mu$. Second, we say that ϕ is "almost realizable" if there is a measurable $E \subset X \times Y$ such that $E_x \in \phi(x)$ for λ-almost all x. Finally, ϕ is "measurable" provided $\phi^{-1}(U)$ is λ-measurable for all open subsets U of the metric space $(\mathscr{E},\rho)$.

This paper arose from a problem raised (orally) by D. Maharam: Characterize the realizable functions ϕ. We give such a characterization in Theorem 3, Corollary, below. In particular, if λ is complete and $\mathscr{E}$ is separable, ϕ is realizable if and only if it is measurable. We also investigate some generalizations and sharpenings of this result.

We remark that, for each measurable $E \subset X \times Y$, E_x is μ-measurable for λ-almost all x. While this is not directly relevant to the question considered here, it does show that the question is a natural one.

We are grateful to the referee for suggestions that have improved the presentation of the results, and in particular for formulating Theorem 3, Corollary.

2. A CHARACTERIZATION OF REALIZABILITY

Let $\mathscr{F}$ be the family of all unions of finitely many "rectangles" $M \times A$, $M \in \mathscr{M}$, $A \in \mathscr{A}$, and let $\mathscr{R}$ be the σ-field generated by $\mathscr{F}$. We shall need the basic measure-theoretic facts summarized in the following theorem.

Received December 9, 1980. Revision received February 18, 1981.
The first author's research was supported by NSF Grant MCS-79-03784.

Michigan Math. J. 28 (1981).

THEOREM 1. *If $B \in \mathscr{B}$, then*
1) *each B_x is μ-measurable,*
2) *the map $f_B : X \to I = [0,1]$ defined by $f_B(x) = \mu(B_x)$ is λ-measurable,*
3) $(\lambda \times \mu)(B) = \int_X \mu(B_x)\, d\lambda(x)$,
4) *the map $\psi_B : X \to \mathscr{E}$ defined by $\psi_B(x) = [B_x]$ is measurable.*

Proof. The assertions 1), 2), 3) are essentially Fubini's theorem; see, for example, [2, Props. 5.1.2 and 5.2.1]. To prove 4), let $\mathscr{C}$ be the family of all $B \in \mathscr{B}$ for which 4) is true. It is easy to see that $\mathscr{C} \supset \mathscr{F}$, and that $\mathscr{C}$ is closed under complementation and under the formation of countable increasing unions. Since $\mathscr{F}$ is a (finitely additive) field, it follows from the "monotone class" theorem (see [3, p. 27]) that $\mathscr{C}$ contains the σ-field generated by $\mathscr{F}$; thus $\mathscr{C} = \mathscr{B}$, as required.

COROLLARY 1. *A set $E \subset X \times Y$ is measurable (with respect to the completed product measure $\lambda \times \mu$) if and only if, for some λ-null $N \in \mathscr{M}$ and some $F \in \mathscr{B}$, $[E_x] = [F_x]$ for all $x \in X - N$.*

For if E is $(\lambda \times \mu)$-measurable, there are sets D and F in $\mathscr{B}$ such that $F \subset E \subset D$ and $(\lambda \times \mu)(D - F) = 0$. Then $\int_X \mu((D - F)_x d\lambda(x) = 0$, so the set

$$N = \{x : \mu(D_x - F_x) \neq 0\}$$

is λ-null (and λ-measurable), and we have $[E_x] = [F_x]$ for all $x \in X - N$. The converse is immediate.

COROLLARY 2. *A map $\phi : X \to \mathscr{E}$ is almost realizable if and only if it is realizable.*

For we can alter an "almost realization" by a suitable null set.

THEOREM 2. *Suppose the measure space $(X, \mathscr{M}, \lambda)$ is complete. If $\phi : X \to \mathscr{E}$ is realizable, then ϕ is measurable.*

Proof. Let E be a realization of ϕ. By Corollary 1 there is $F \in \mathscr{B}$ such that $\phi(x) = \psi_F(x)$ for λ-almost all x. Since ψ_F is measurable, by Theorem 1, it follows that ϕ is measurable.

Definition. We say that $\phi : X \to \mathscr{E}$ is "essentially separable-valued" if there is a λ-null set $N \subset X$ such that $\phi(X - N)$ is a separable subset of $(\mathscr{E}, \rho)$.

THEOREM 3. *If $\phi : X \to \mathscr{E}$ is realizable, then ϕ is essentially separable-valued. Conversely, if ϕ is measurable and essentially separable-valued, then ϕ is realizable.*

Proof. With the notation of the proof of Theorem 1, let $\mathscr{W}$ be the family of all $B \in \mathscr{B}$ such that the set $S(B) = \{[B_x] : x \in X\}$, of the measure-classes of the sections of B, is separable. Then $\mathscr{W} \supset \mathscr{F}$; in fact, if $F \in \mathscr{F}$ then $S(F)$ is finite, hence separable. $\mathscr{W}$ is closed under complementation, because complementation induces an isometry of $(\mathscr{E}, \rho)$ onto itself. Finally, $\mathscr{W}$ is closed with respect to countable increasing unions. For if $B = \cup \{B_n : n \in \mathbf{N}\}$, where $B_1 \subset B_2 \subset \ldots$ and each $B_n \in \mathscr{W}$, let D_n be a countable dense subset of $S(B_n)$ and put

$$D = \cup \{D_n : n \in N\}.$$

It is easy to see that $S(B) \subset \bar{D}$; hence, as a subset of a separable metric space, $S(B)$ is separable. Thus, as in the proof of Theorem 1, it follows that $\mathscr{W} = \mathscr{B}$.

Now let E be any realization of ϕ. By Theorem 1, Corollary 1, we have $[E_x] = [B_x]$ for all $x \in X - N$, where N is a λ-null set and $B \in \mathscr{B}$. Thus $\phi(X - N) \subset S(B)$ and is separable.

Conversely, suppose that ϕ is measurable and that $\phi(X - N)$ is separable, where N is λ-null. Let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of $\phi(X - N)$, and choose for each n a set D_n in the measure-class d_n. For $n = 1, 2, \ldots$ define $\psi_n : X \to \mathscr{E}$ by: $\psi_n(x) = d_1$ if $x \in N$, and $\psi_n(x) =$ first d_m such that

$$\rho(\phi(x), d_m) < 1/n \quad \text{if} \quad x \in X - N.$$

Thus each ψ_n is measurable and countable-valued, and

$$\rho(\phi(x), \psi_n(x)) < 1/n \quad \text{for all } x \in X - N.$$

Now let $C_{nm} = \psi_n^{-1}(d_m)$, and put $E(n) = \bigcup_m (C_{nm} \times D_m)$ It is easy to see that $E(n)$ is a realization of ψ_n, and thence that $\bigcap_n \bigcup_{k \geq n} E(2^k)$ is an "almost" realization of ϕ. By Corollary 2, ϕ is realizable.

COROLLARY. *Suppose the measure space* $(X, \mathscr{M}, \lambda)$ *is complete. Then the following statements about a map* $\phi : X \to \mathscr{E}$ *are equivalent:*

(i) ϕ *is realizable,*

(ii) ϕ *is almost realizable,*

(iii) ϕ *is measurable and essentially separable-valued.*

3. FROM MEASURABILITY TO REALIZABILITY

It follows from Theorem 3 that if $\mathscr{E}$ is separable (equivalently, if $\mathscr{E}$ is countably generated) then the measurability of $\phi : X \to \mathscr{E}$ implies its realizability. We next show that here the separability of $\mathscr{E}$ can be replaced by a set-theoretic hypothesis on X. First, some definitions. A cardinal κ is said to be "real-valued measurable" if there is a countably additive probability measure defined on all subsets of κ and vanishing on all singletons. By the "cellularity" of $(X, \mathscr{M}, \lambda)$ is meant the smallest cardinal number m such that every sub-family of $\mathscr{M}$ of pairwise disjoint nonempty sets has cardinality $\leq m$. Note that this cellularity is at most the cardinality of X.

THEOREM 4. *Suppose there is no real-valued measurable cardinal* $\leq$ *cellularity of* $(X, \mathscr{M}, \lambda)$. *If* ϕ *is measurable, then* ϕ *is realizable.*

Proof. Let $\epsilon > 0$ be fixed for the moment. Take a σ-discrete open refinement of the open cover of $(\mathscr{E}, \rho)$ by $\epsilon/2$-spheres, say $\mathscr{U} = \cup \{\mathscr{U}_n : n \in \mathbf{N}\}$, where $\mathbf{N}$ is the set of positive integers and where, for each n, $\mathscr{U}_n$ is discrete; say $\mathscr{U}_n = \{U_{n\alpha} : \alpha \in \Lambda_n\}$. Put $B_{n\alpha} = \phi^{-1}(U_{n\alpha})$, and note that every union of sets $B_{n\alpha}$

is measurable, because $\cup\{B_{n\alpha}:\alpha\in A\}=\phi^{-1}(\cup\{U_{n\alpha}:\alpha\in A\})$, the inverse of an open set. For each n, put $M_n=\{\alpha\in\Lambda_n:\lambda(B_{n\alpha})>0\}$. Thus M_n is countable. Also $\lambda(\cup\{B_{n\alpha}:\alpha\in\Lambda_n-M_n\})=0$, since otherwise one could choose a point x_α from each nonempty $B_{n\alpha}$ and construct a nontrivial countably additive measure τ, defined on all subsets of $Z=\{x_\alpha:\lambda(B_{n\alpha})=0\}$ and vanishing on singletons, by setting, for each $W\subset Z$, $\tau(W)=\lambda(\cup\{B_{n\alpha}:x_\alpha\in W\})$.

Put $S(\epsilon)=\bigcup_n(\cup B_{n\alpha}:\alpha\in M_n)$; thus $\lambda(X-S(\epsilon))=0$ and $\phi(S(\epsilon))$ has a countable covering by sets of diameter $<\epsilon$. Finally, put $S=\cap\{S(1/n):n\in\mathbf{N}\}$; then $\lambda(X-S)=0$ and $\phi(S)$ is separable. So ϕ is realizable, by Theorem 3.

COROLLARY. *It is relatively consistent with ZFC that the measurability of* ϕ *implies the realizability of* ϕ.

The proof of Theorem 4 can easily be modified to prove the following theorem.

THEOREM 5. *Suppose the density character of* $(\mathscr{E},\rho)$ *is not real-valued measurable. If* ϕ *is measurable, then* ϕ *is realizable.*

COROLLARY. *Suppose* $(\mathscr{E},\rho)$ *is separable, or more generally has density character less than the first weakly inaccessible cardinal. If* ϕ *is measurable, then* ϕ *is realizable.*

We shall now show that the assumptions, in Theorems 4 and 5, about real-valued measurable cardinals cannot be eliminated.

THEOREM 6. *Suppose there is a real-valued measurable cardinal. Then there are measure spaces* $(X,\mathscr{M},\lambda)$ *and* $(Y,\mathscr{A},\mu)$*, with complete probability measures, and a measurable* $\phi:X\to\mathscr{E}$ *that is not realizable.*

Here $\mathscr{E}$, as before, is the measure algebra of Y, metrized by

$$\rho([A],[B])=\mu(A\triangle B).$$

Proof. Let κ be the smallest real-valued measurable cardinal. Take X to be a set of cardinal κ, and let λ be the corresponding measure (a probability measure, defined for all subsets of X, and vanishing on singletons). Take $(Y,\mathscr{A},\mu)$ to be the measuretheoretic product $\prod\{I_\alpha:\alpha<\kappa\}$ of κ copies of the unit interval I, with product Lebesgue measure. There is a family $\mathscr{D}$ consisting of κ measure classes in $\mathscr{E}$, every two of which are at ρ-distance $1/2$ (for instance, the classes $\left[\prod J_\alpha:\alpha<\kappa\right]$ for which one J_α is $[0,1/2]$ and the others are I). Take ϕ to be a one-to-one function from X onto $\mathscr{D}$. Then ϕ is measurable, since every subset of X is measurable.

Suppose ϕ has a realization, E. By Theorem 3 there is a λ-null set N such that $\phi(X-N)$ is separable. But $X-N$ has cardinal κ, because the restriction of λ to $X-N$ is still a probability measure, defined on all its subsets and vanishing on singletons. So $\phi(X-N)$ is a subset of $\mathscr{D}$ of cardinal κ, and cannot be separable—a contradiction.

COROLLARY. *The following are equivalent:*

(i) *Every measurable* ϕ *is realizable.*

(ii) *There are no real-valued measurable cardinals.*

4. THE BOREL CASE

Finally we consider the situation in which the maps and realizations are required to be Borel. Thus we suppose henceforth that λ, μ are Borel measures, so that $\mathscr{M}$ and $\mathscr{A}$ consist of the Borel subsets of X, Y, respectively.

THEOREM 7. *Suppose that* X *and* Y *are metric spaces, and that* Y *is separable. Then a map* $\phi: X \to \mathscr{E}$ *is Borel measurable if, and only if,* ϕ *has a Borel realization.*

Proof. First, if ϕ is Borel measurable, then (using the fact that here $(\mathscr{E},\rho)$ is separable and hence has a countable base) one can easily modify the proof of Theorem 3 to obtain a Borel subset E of $X \times Y$ such that, for all $x \in X$, $E_x \in \phi(x)$.

Second, suppose E is a Borel subset of $X \times Y$ that realizes ϕ. Each open subset U of $\mathscr{E}$ is a countable union of sets of the form $U_\epsilon(A) = \{e \in \mathscr{E} : \rho(e,[A]) < \epsilon\}$, where $\epsilon > 0$ and A is a Borel subset of Y. It will suffice to prove that each $\phi^{-1}(U_\epsilon(A))$ is Borel measurable. Let $h(x) = \mu(E_x \triangle A)$, for each $x \in X$. It is well known (and easy to check) that h is a Borel measurable function. Thus $\phi^{-1}(U_\epsilon(A)) = h^{-1}([0,\epsilon))$ is a Borel set; and the proof is complete.

Problem 1. Is there a "reasonable" extension of Theorem 7 to nonseparable Y?

When X and Y are complete separable metric spaces, the "only if" implication in Theorem 7 can be sharpened. First, a theorem from descriptive set theory.

THEOREM 8. *If* X *and* Y *are uncountable complete separable metric spaces, then there is a coanalytic subset* N *of* $X \times Y$ *which is universal for the Borel subsets of* Y *that have* μ*-measure zero.*

Proof. Take a coanalytic subset C of $X \times Y$ that is universal for the Borel subsets of Y—i.e., each C_x $(x \in X)$ is a Borel subset of Y, and if B is a Borel subset of Y then $B = C_x$ for some $x \in X$.

Let $S = \{x : \mu(C_x) > 0\}$. It follows from Theorem 4.1 of [1] that S is a coanalytic subset of X. Let f be a Borel measurable map of X onto $X - S$, and let

$$N = (f \times id)^{-1}(C).$$

Then N is a coanalytic subset of $X \times Y$ having the required properties.

THEOREM 9. *Let* X *and* Y *be uncountable complete separable metric spaces, and let* $\phi: X \to \mathscr{E}$ *be a Borel measurable map. There is a subset* R *of* $X \times X \times Y$ *that is the difference of two analytic sets and is such that*

(i) *for each* $t \in X$, $R_t = \{(x,y) : (t,x,y) \in R\}$ *is a Borel realization of* ϕ,

(ii) *for each* $x \in X$, $\{R_{(t,x)} : t \in X\}$ *is the family of all Borel sets in* $\phi(x)$, *where* $R_{(t,x)} = \{y : (t,x,y) \in R\}$.

Proof. Let τ be a Borel isomorphism of X onto $X \times X$; say $\tau(x) = (\tau_1(x),\tau_2(x))$.

For each $i = 1, 2$, let $h_i(t,x,y) = (t,\tau_i(x),y)$. Let $N_i = h_i^{-1}(X \times N)$, where N is a coanalytic subset of $X \times Y$ as in Theorem 8. Notice that N_1 and N_2 are coanalytic subsets of $X \times X \times Y$, and if B_1 and B_2 are Borel subsets of Y with μ-measure zero then there is some $(t,x) \in X \times X$ so that $(N_i)_{(t,x)} = B_i$ for $i = 1, 2$.

Let E be a Borel realization of ϕ. Put $R = ((X \times E) - N_1) \cup N_2$. Then the set R has all the required properties.

The following question remains unanswered.

Problem 2. Let X and Y be uncountable complete separable metric spaces, with Borel measures λ, μ respectively. Let ϕ be a Borel measurable map of X into $\mathscr{E}$. Is there an absolutely measurable (or $\lambda \times \lambda \times \mu$-measurable) subset W of $X \times X \times Y$ such that (1) for each $t \in X$, W_t is a Borel realization of ϕ, and (2) if E is a Borel realization of ϕ then there is some $t \in X$ so that $W_t = E$?

Added in Proof: The authors are grateful to S. D. Chatterji and M. P. Ershov for pointing out that the problem considered here is known to probabilists under the name "measurable modification," and for supplying the following references to earlier work:

Donald L. Cohn, *Liftings and the construction of stochastic processes.* Trans. Amer. Math. Soc. 246 (1978), 429–438. J. Hoffman-Jørgensen, *Existence of measurable modifications of stochastic processes.* Z. Wahrsch. Verw. Gebiete 25 (1973), 205–207.

REFERENCES

1. D. Cenzer and R. D. Mauldin, *Inductive definability: measure and category.* Adv. in Math. 38 (1980), no. 1, 55-90.
2. Donald L. Cohn, *Measure theory,* (Birkhäuser), Boston, Mass., 1980.
3. P. R. Halmos, *Measure theory,* D. Van Nostrand, New York, 1950.

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