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REPRESENTATIONS OF WELL-FOUNDED PREFERENCE ORDERS

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A preference order, or linear preorder, on a set X is a binary relation $\preceq$ which is transitive, reflexive and total. This preorder partitions the set X into equivalence classes of the form $[x] = \{y:x \preceq y \text{ and } y \preceq x\}$. The natural relation induced by $\preceq$ on the set of equivalence classes is a linear order. A well-founded preference order, or prewellordering, will similarly induce a well-ordering. A representation or Paretian utility function of a preference order is an order-preserving map f from X into the $\mathbf{R}$ of real numbers (provided with the standard ordering). Mathematicians and economists have studied the problem of obtaining continuous or measurable representations of suitably defined preference orders [**4**, **7**]. Parametrized versions of this problem have also been studied [**1**, **7**, **8**]. Given a continuum of preference orders which vary in some reasonable sense with a parameter t, one would like to obtain a continuum of representations which similarly vary with t.

Specifically, let T and X be Polish (that is, complete separable metric) spaces. For each t in T, let B_t be a nonempty subset of X, $\preceq_t$ a preference order on B_t and let $E_t = \{(x, y):x \preceq_t y\}$. Finally, set

$$E = \{(t, x, y):x \preceq_t y\},$$

and set

$$B = \{(t, x):x \in B_t\}.$$

Suppose that E (and therefore B) is a Borel measurable set. This will be the general setting throughout the paper.

We will say that E is *section-wise closed* if, for each t, E_t is closed with respect to $B_t \times B_t$; in this case, each preference order $\preceq_t$ will possess a continuous representation by a result of Debreu [**4**]. The second author

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showed in [7] that if E is section-wise closed, then there is an $S(T \times X)$-measurable map f of B into $\mathbf{R}$ such that; for each t, $f(t, \cdot)$ is a continuous representation of $\preceq$. (Here $S(T \times X)$ form the C-sets of Selivanovskii or the husin heirarchy [**6**, p. 468].) Under the further assumption that each B_t is σ-compact, it was also shown in [7] that the map f may be taken to be Borel measurable.

In this paper we obtain significant improvements in the above results in the case that each preference order $\preceq_t$ is well-founded.

THEOREM 3.3. *Let E be a Borel subject of the product $T \times X \times X$ of Polish spaces such that, for each t, $E_t = \{x, y):(t, x, y) \in E\}$ is a well-founded preference order on $B_t = \{x:(t, x, x) \in E\}$. Then there is a Borel measurable map f from $B = \{ (x, t):x \in B_t\}$ into $\mathbf{R}$ such that each $f(t, \cdot)$ is a representation of E_t.*

If E is section-wise closed, then we show that the map f constructed in the above theorem can be modified so as to be continuous on each section.

THEOREM 4.2. *Suppose that E satisfies the hypothesis of Theorem* 3.1 *and that, for each t, E_t is closed with respect to $B_t \times B_t$. Then there is a Borel measurable map f from B into* $\mathbf{R}$ *such that each $f\ (t, \cdot)$ is a continuous representation of E_t.*

This answers Question (2) of [7] in the affirmative.

We note that the methods of this paper are quite different from those of [7]. The construction of the map f in Theorem 3.3 does not require that E be section-wise closed and does not depend on any selection principles.

1. Ordinal representations. In this section, we introduce the notion of an ordinal representation of a preference order and of a continuum of preference orders. We show the existence of ordinal representations for individual well-founded preference orders and give a sufficient condition for the continuity of such a representation. Finally, we show that the existence of an ordinal representation (with range bounded to some countable ordinal) for a continuum of preference orders implies the existence of a representation into the real line.

An ordinal representation of a preference order $\preceq$ on a set B is simply an order-preserving map ϕ from B into the class of ordinal numbers. Suppose now that $\preceq$ is a well-founded Borel preference order on a Borel subset B of a Polish space X. Then $\preceq$ possesses a natural ordinal representation, which we will now describe. Let $x \sim y$ denote the equivalence relation ($x \preceq$ y and $y \preceq x$) and let $x \preceq y$ denote ($x \preceq y$ and not $(y \preceq x)$). Let the ordinal $o(\preceq) = \kappa$ be the order type of the induced well-ordering on the equivalence classes of $\sim$; it follows from Theorem 3.1 that κ is countable. For $X \in B$, let $o(x)$ be the order type of $\preceq$ restricted to the predecessors of $[x]$. Note that

$$o(\preceq) = \sup \{o(x) + 1 : x \in B\}.$$

The map $o: B \to (\preceq)$ is clearly an ordinal representation.

Furthermore, since each equivalence class $[x]$ is a Borel subset of X, $o^{-1}(A)$ will be Borel for any set A of ordinals. This will clearly apply to any representation of a Borel preference order.

Let the class of ordinals be given the usual order topology with a subbase of open sets of the two forms $\{\alpha:\alpha < \beta\}$ and $\{\alpha:\alpha > \beta\}$. If $\preceq$ has a continuous representation ϕ, then, for each $y \in B$, both $\{x:x \preceq y\} = \{x: \phi(x) \leqq \phi(y)\}$ and $\{x \geqq y\}$ must be relatively closed subsets of B. A preference order satisfying the above condition was said to be continuous in [7]. This condition is easily seen to be equivalent to the following: that the set $E = \{(x, y):x \preceq y\}$ is a relatively closed subset of $B \times B$.

LEMMA 1.1. *An ordinal representation ϕ of a continuous preference order on B is continuous if and only if, for each ordinal β, $\{x:\phi(x) > \beta\}$ is a relatively open subset of B.*

Proof. For any ordinal β, $\{x:\phi(x) < \beta\}$ equals either B or $\{x:x \preceq y\}$, where $\phi(y)$ is the least ordinal in the range of ϕ which is greater than or equal to β.

For a continuous preference order $\preceq$, the map o defined above is a continuous ordinal representation, since, for each ordinal β, $\{x:o(x) > \beta\}$ equals either $\emptyset$ or $\{x:x \succ y\}$, where $o(y) = \beta$.

We will next indicate how (continuous) ordinal representations may be used to obtain (continuous) representations into the real line. It is a classical result of Cantor that any countable linear ordering can be imbedded into the real line. For a well-ordering, the image can be taken to be a closed set. This fact is a straightforward consequence of the countable axiom of choice.

LEMMA 1.2. *For any countable ordinal κ, there exists a bicontinuous order isomorphism i of $\kappa = \{\alpha:\alpha < \kappa\}$ onto a closed subset K of the real line.*

It should be remarked that any order isomorphism i from an initial segment κ of the ordinals onto a closed set of reals must be bicontinuous. This can be seen as follows. For any real r, $\{\alpha:i(\alpha) < r\} = \{\alpha:\alpha < \beta\}$, where β is either κ or the least such that $i(\beta) \geqq r$; also, $\{\alpha:i(\alpha) \leqq r\}$ is either empty or equals $\{\alpha:\alpha \leqq \beta\}$, where $i(\beta)$ is the least upper bound of K

$\cap(-\infty\ r]$. The inverse map from K onto κ is just the natural representation of the standard order on K and is therefore continuous as shown above.

Now if $\preceq$ is a continuous preference order on B, let o be the natural ordinal representation mapping B onto $o(\preceq) = \kappa$ and let i be a continuous order isomorphism of κ onto a closed subset K of the real line. Then the composition of $f:B \to K$, defined by $f(x) = i(o(x))$ is clearly a continuous representation of B into the real line.

The problem is more interesting when we are given a continumum of preference orders. Therefore, let the Borel subset E of the product $T \times X \times X$ of Polish spaces define a continuum of preference orders $\preceq_t$ on the sets B_t as described in the introduction. An ordinal representation of E is a map ϕ from B into the class of ordinals such that, if x and y belong to B_t, then $x \preceq_t y$ if and only if $\phi(t, x) \leqq \phi(t, y)$. It is important to note that the natural map ϕ, defined by letting $\phi(t, \cdot)$ be the natural ordinal representation of B_t, is not necessarily a Borel map, even assuming that E is section-wise closed. An example will be given in Section two. The failure of this natural first guess for a Borel representation of a continuum of preference orders necessitates the inductive construction given in this paper.

However, once we construct a continuous or Borel ordinal representation for E which maps B into some countable ordinal κ, Lemma 1.2 can be used to obtain a continuous or Borel representation mapping B into the real line.

2. Reduction, separation and boundedness. The classical Separation Theorem of Lusin states that disjoint analytic subsets A_1 and A_2 of a Polish space Y may be separated by a Borel set D so that $A_1 \subset D$ and $A_2 \cap D = \emptyset$. Now suppose that $\sim$ is an analytic equivalence relation on Y, that is, $\{(x, y): x \sim y\}$ is an analytic subset of $Y \times Y$; in fact, we have in mind the equivalence relation on the product space $T \times X$ induced by a Borel continuum of preference orders $\preceq_t$. Define the saturation $S(A)$ of a subset A of Y by

$$S(A) = \{x:(\exists\, y \in A)\ x \sim y\}.$$

Of course, the saturation of an analytic set is also analytic. We will need the "invariant" separation theorem first obtained by Ryll-Nardzewski and a "downward closed invariant" reduction theorem.

THEOREM 2.1. (Invariant Separation) *Let $\sim$ be an analytic equivalence relation on a Polish space Y. Then any two disjoint saturated analytic subsets A_1 and A_2 of Y may be separated by a saturated Borel set D.*

The Reduction Theorem of Kuratowski [**6**, p. 508] for a infinite sequence $\{C_1, C_2, \dots\}$ of coanalytic sets whose union is Borel states that there exists a sequence $\{D_1, D_2, \dots\}$ of pairwise disjoint Borel sets such that $D_n \subset C_n$ for each n and $\cup\, C_n = \cup\, D_n$. Now, if each C_n is saturated, then $S(D_n)$ and $Y\text{-}C_n$ are disjoint, saturated analytic sets. Thus, by the Invariant Separation Theorem above, there exists a Borel B_n such that $D_n \subset S(D_n) \subset B_n \subset C_n$. This gives the first part of Theorem 2.2.

THEOREM 2.2. (Invariant Reduction) *Let $\sim$ be an analytic equivalence relation on a Polish space Y and let $\{C_0, C_1, C_2, \dots\}$ be a sequence of saturated coanalytic subsets of Y such that $\cup_n C_n = D$ is Borel. Then there exists a sequence $\{B_n : n < \omega\}$ of saturated Borel sets such that $B_n \subset C_n$ for all n and such that $\cup_n B_n = D$. Furthermore, if $\preceq$ is a Borel linear ordering on the equivalence classes of $\sim$ and each C_n is closed downward, then each B_n may be taken to be closed downwards.*

Proof. The proof of the first part was given above. Now fix n and suppose that $C_n = C$ is closed downward. Let the saturated Borel subset $B_n = B^{\circ}$ of C be given by the above and let

$$A^{\circ} = \{y : (\exists\, x \in B^{\circ})(y \preceq x)\}.$$

Then A° is a saturated analytic subset of C, so by Theorem 2.1, there is a saturated Borel set B^1 with $A^{\circ} \subset B^1 \subset C$. Proceeding inductively, we obtain a sequence $B^{\circ} \subset A^{\circ} \subset B^1 \subset A^1 \subset \dots$ of saturated subsets of C such that each B^i is Borel and each A^i is analytic and closed downwards. Then $\cup_i B^i$ will be Borel, saturated and closed downwards.

The invariant separation and reduction theorems are both subsumed under the main result of [**2**].

The classical Boundedness Principle of Lusin and Sierpinski states that any analytic subset of the family of countable well-orderings must be bounded in length by some countable ordinal. This can be used to see that a Borel continuum of well-founded preference orders is similarly bounded in length.

We will use the Boundedness Principle as incorporated in the Inductive Definability Theorem of [**3**]. We recall that a monotone operator over the Polish space Y is a map Γ from the power set 2^Y into 2^Y such that, whenever $K \subset M \subset Y$, $\Gamma(K) \subset \Gamma(M)$. Γ constructs a transfinite sequence $\{\Gamma^{\alpha} : \alpha \text{ an ordinal}\}$ by letting $\Gamma^{\circ} = \emptyset$, $\Gamma^{\alpha+1} = \Gamma(\Gamma^{\alpha})$ for all α and $\Gamma^{\lambda} = \cup_{\alpha<\lambda} \Gamma^{\alpha}$ for limit λ.

The closure $\mathrm{Cl}(\Gamma) = \Gamma^{\infty}$ of Γ is $\cup_{\alpha} \Gamma^{\alpha}$; the closure ordinal $|\Gamma|$ is the least such that $\Gamma^{\alpha} = \Gamma^{\infty}$. The following theorem is given in [**3**, p. 58].

THEOREM 2.3. (Inductive Definability) *Let* Γ *be a coanalytic monotone operator on a Polish space Y. Then*

(a) *For each countable ordinal* α, Γ^{α} *is a coanalytic subset of Y.*

(b) Γ^{∞} *is a coanalytic subset of Y.*

(c) $|\Gamma| \leqq \omega_1$.

(d) *For any analytic subset A of* Γ^{∞}, *there is a countable ordinal* α *such that* $A \subset \Gamma^{\alpha}$.

Part (d) can be viewed as a generalization of the Boundedness Principle.

3. Borel representations. Let E be a Borel continuum of well-founded preference orders $\preceq_t$ on the Borel subset B of the product $T \times X$ of Polish spaces, as described in the introduction. For each t, let $o(t)$ be the order type of the induced well-ordering on the equivalence classes of $\sim_t$; for each x, let $o(x, t)$ be the order type of $\preceq_t$ restricted to the $\preceq_t$-predecessors of $[x]_t$. Let $o(E) = \sup_t o(t)$.

THEOREM 3.1. *Let E be a Borel subset of the product* $T \times X \times X$ *of Polish spaces such that, for each t,* $E_t = \{(x, y):(t, x, y) \in E\}$ *is a well-founded preference order on* B_t. *Then* $o(E)$ *is countable and each of the following sets are coanalytic*:

$$\{(t, x):o(t, x) < \alpha\}, \quad \{(t, x):o(t, x) \leqq \alpha\},$$
$$\{t:o(t) \leqq \alpha\}, \text{ and } \quad \{t:o(t) < \alpha\}.$$

Proof: Define the $\prod_1^1$ monotone operator Γ over B by:

$$(t, x) \in \Gamma(K) \Leftrightarrow (\forall y)[(y \preceq_t x) \rightarrow (t, y) \in K].$$

It is easily seen by induction on α that

$$\Gamma^{\alpha} = \{(t, x):o(t, x) < \alpha\};$$

in addition, $\mathrm{Cl}(\Gamma) = B$ and $|\Gamma| = o(E)$.

Γ^{α} is $\prod_1^1$ by Theorem 2.3(a). Also,

$$\begin{aligned} o(t, x) &\leqq \alpha \Leftrightarrow o(t, x) < \alpha + 1 \\ o(t) &\leqq \alpha \Leftrightarrow (\forall x)\, o(t, x) < \alpha; \\ o(t) &< \alpha \Leftrightarrow (\exists \beta < \alpha)\, o(t) \leqq \beta. \end{aligned}$$

Now by Theorem 2.3(d), $B = \mathrm{Cl}(\Gamma) = \Gamma^\alpha$ for some countable ordinal α; it follows that $o(E)$ is countable.

We are now ready for the first of our two main theorems.

THEOREM 3.2. *Let E be a Borel continuum of well-founded preference orders on a subset B of $T \times X \times X$ as described in Theorem* 3.1. *Then E possesses a Borel ordinal representation $\phi: B \to o(E)$.*

Proof. The proof is by induction on $\alpha = o(E)$.

($\alpha = 1$). Just let $\phi(t, x) = 0$ for all (t, x) in B.

($\alpha + 1$). Suppose the theorem holds for $o(E) = \alpha$ and let B, E be given with $o(E) = \alpha + 1$.

Let

$$U = \{ (t, x) \in B : o(t, x) \geqq \alpha\} \quad \text{and}$$
$$L = \{ (t, x) \in B : (\exists\, y)\, x \prec_t y\}.$$

Then U and L are disjoint saturated analytic subsets of B. By the Invariant Separation Theorem (2.1), there exist disjoint saturated Borel sets $B_L \supset L$ and $B_U \supset U$ such that $B_L \cup B_U = B$. Define a Borel continuum E_L of well-founded preference orders on B_L by

$$E_L = E \cap \{ (t, x, y) : \quad (t, x) \in B_L \quad \text{and} \quad (t, y) \in B_L\}.$$

Now $o(E_L) = \alpha$, so by the induction hypothesis, E_L possesses a Borel ordinal representation $\phi_L : B_L \to \alpha$. Define the representation ϕ of E by:

$$\phi(t, x) = \begin{cases} \alpha \text{ if } (t, x) \in B_U, \\ \phi_L\,(t, x) \text{ if } (t, x) \in B_L. \end{cases}$$

Each $\phi^{-1}(\{\beta\})$ is either B_U, $\emptyset$, or $\phi_L{}^{-1}(\{\beta\})$ and is therefore a Borel subset of B. If (t, x) and (t, y) are both in B_L, then

$$x \preceq_t y \Leftrightarrow \phi_L\,(t, x) \leqq \phi_L\,(t, y) \Leftrightarrow \phi(t, x) \leqq \phi(t, y).$$

If (t, x) and (t, y) are both in B_U, then $x \sim_t y$ and $\phi(t, x) = \phi\,(t, y) = \alpha$. Finally, if $(t, x) \in B_L$ and $(t, y) \in B_U$, then $(t, y) \notin L$, so for all $z \in B_t$, $z \preceq_t y$; it follows that $x \preceq_t y$. Since $(t, x) \notin B_U$ and B_U is saturated, we must have $x \preceq_t y$. Of course

$$\phi(t, x) = \phi_L(t, x) < \alpha = \phi(t, y).$$

Thus ϕ is an ordinal representation.

(λ = limit). Let $\lambda = \lim_n(\alpha_n)$, where $\{\alpha_n : n < \omega\}$ is an increasing sequence and the theorem holds for each ordinal $\alpha < \lambda$. Suppose that $o(E) = \lambda$. For each n, let

$$C_n = \{ (t, x): o(t, x) < \alpha_n \}.$$

Then each C_n is Π_1^1 and saturated (in fact, closed downwards). Furthermore, each $C_n \subset C_{n+1}$ and $\cup_n C_n = B$. By the Invariant Reduction Theorem (2.2), there is a sequence $\{B_n : n < \omega\}$ of saturated Borel sets such that $\cup B_n = B$ and, for each n, $B_n \subset C_n$ and B_n is closed downwards.

$$\text{Let } E_n = E \cap \{ (t, x, y): (t, x) \in B_n \quad \text{and} \quad (t, y) \in B_n \}.$$

Let $o_n(t, x)$ be the order of (t, x) in E_n, $o_n(t)$ the order of $B_{n,t}$ and $\tau_n = o(E_n)$. Note that $\tau_n \leqq \alpha_n$. By the induction hypothesis, each E_n possesses a Borel ordinal representation $\phi_n : B_n \to \tau_n$. Define the map $\phi : E \to \lambda$ by

$$\phi(t, x) = \min \{ \phi_n(t, x) : (t, x) \in B_n \}.$$

Then for each ordinal β, we have

$$\phi(t, x) > \beta \Leftrightarrow (\forall n)((t, x) \in B_n \to \phi_n(t, x) > \beta)$$

and

$$\phi(t, x) < \beta \Leftrightarrow (\exists n)((t, x) \in B_n \quad \text{and} \quad \phi_n(t, x) < \beta).$$

It follows that ϕ is Borel measurable. Now, given (t, x) and (t, y) in B, such that $x \preceq_t y$, choose n so that $(y, t) \in B_n$ and $\phi_n(y, t) = \phi(y, t)$. Since B_n is closed downward, $(t, x) \in B_n$ and since ϕ_n is a representation,

$$\phi_n(t, x) \leqq \phi_n(t, y) = \phi(t, y).$$

But this implies that $\phi(t, x) \leqq \phi(t, y)$ since $\phi(t, x)$ is the minimum of the $\phi_n(t, x)$. Similarly, if $x \preceq_t y$, then $\phi(t, x) < \phi(t, y)$. This completes the proof of Theorem 3.2.

THEOREM 3.3. *Let E be a Borel continuum of well-founded preference orders on B as described in Theorem 3.1. Then E possesses a Borel representation $f: B \to R$.*

Proof. Let $\phi : B \to o(E) = \kappa$ be given by Theorem 3.2 and let $i : \kappa \to K$ be given by Lemma 1.2. Define f by $f(x) = i(\phi(x))$.

One may wonder why we don't dispense with reduction and separation and just let $\phi(t, x) = o(t, x)$. The following example indicates that this may not be possible even when $o(E) = 2$ and each E_t is continuous. Let T and X be the space of irrational numbers, let S be an analytic non-Borel subset of T, let $A = S \times \{0\} \cup T \times \{1\}$ and let f be a continuous map of X onto A. Now $f(x) = (f_1(x), f_2(x))$, where both f_1 and f_2 are continuous. Define the closed subset B of $T \times X$ by

$$B = \{ (t, x): f_1(x) = t \}$$

and the closed subset E of $T \times X \times X$ by

$$E = \{ (t, x, y): f_1(x) = f_1(y) = t \quad \text{and} \quad f_2(x) \leqq f_2(y) \}.$$

Also define the closed sets

$$B_i = \{ (t, x) \in B: f_2(t, x) = i \} \quad \text{for } i = 0 \text{ or } 1.$$

Of course, the map f_2 is a continuous representation of E, but it does not always agree with the order map $o(t, x)$. In fact, let

$$C_0 = \{ (t, x): o(t, x) = 0 \}.$$

Then

$$C_0 = B_0 \cup [B_1 \cap ((T - S) \times X)].$$

If C_0 were Borel, then $C_0 \cap B_1 = (T - S) \times X$ would also be Borel, whereas it is clearly a coanalytic non-Borel set by our choice of S.

4. Continuous representations. Suppose that we have a Borel representation $\phi: B \to o(E)$ for a continuum E of continuous well-founded preference orders. We will now systematically repair any discontinuities of ϕ and thus obtain a section-wise continuous representation of E.

THEOREM 4.1. *Suppose that E is section-wise closed and that $\phi: B \to o(E)$ is a Borel representation of E. Then E possesses a section-wise continuous Borel representation $\bar{\phi}: B \to o(E) = \kappa$.*

Proof. We will construct a descreasing sequence $\{\phi_\alpha: \alpha \leqq \kappa\}$ of Borel representations of E such that $\phi_0 = \phi$ and, for all $\alpha \leqq \kappa$,

(1) for all $t \in T$ and all $\sigma < \alpha$:

$$\{x: \phi_\alpha(t, x) > \sigma\} \text{ is open in } B_t.$$

(2) for all $(t, x) \in B$ and all $\sigma < \beta < \alpha$:

$$\phi_\beta(t, x) > \sigma \Leftrightarrow \phi_\alpha(t, x) > \sigma.$$

The map $\bar{\phi} = \phi_\kappa$ will be a Borel ordinal representation which is section-wise continuous by (1) and Lemma 1.1. The construction of the maps ϕ_α is by induction and as usual, there are two cases to consider: successor and limit.

(Case I: $\alpha + 1$) Suppose that ϕ_β has been constructed, satisfying (1) and (2), for all $\beta \leqq \alpha$. Define the saturated coanalytic subset C of B by

$$\begin{aligned} C &= \{ (t, x) : \sup \{\phi_\alpha(t, y) + 1 : y \prec_t x\} \leqq \alpha < \phi_\alpha(t, x) \} \\ &= \{ (t, x) : \phi_\alpha(t, x) > \alpha \quad \text{and} \quad (\forall\, y)(y \prec_t x \rightarrow \phi_\alpha(t, y) < \alpha\}. \end{aligned}$$

Define the analytic set A which is a subset of C by

$$A = \{ (t, x) : \phi_\alpha(t, x) > \alpha \quad \text{and} \quad (\forall\, n)(\exists\, y)(d(x, y) < \frac{1}{n} \quad \text{and} \quad \phi_\alpha(t, y) \leqq \alpha) \},$$

where d is the metric on X.

Now A contains precisely those points of C at which ϕ_α is discontinuous because of the indicated gap: $\sup \{\phi_\alpha(t, y) : y \prec_t x\} \leqq \alpha$ whereas $\phi_\alpha(t, x) > \alpha$. Notice that in fact if $(t, x) \in A$, then $\sup \{\phi_\alpha(t, y) : y \prec_t x\}$ must equal α. To see this, suppose

$$\sup \{\phi_\alpha(t, y) : y \prec_t x\} = \beta < \alpha.$$

Since $(t, x) \in A$, there is a sequence $\{y_n : n < \omega\}$ converging to x such that $\phi_\alpha(t, y_n) < \alpha$ for each n. Since $\phi_\alpha(t, x) > \alpha$ and ϕ_α is a representation, $y_n \prec_t x$ for each n. Now, according to (1), $U = \{y : \phi_\alpha(t, y) > \beta\}$ is open in B_t. Since $x \in U$, it follows that for some n, $y_n \in U$ and therefore $\phi_\alpha(t, y_n) > \beta$. This is a contradiction.

Thus we can repair ϕ_α for $(t, x) \in A$ letting $\phi_{\alpha+1}(t, x) = \alpha$. For $(t, x) \in C - A$,

$$\{y : x \preceq_t y\} = \{y : \phi_\alpha(t, y) > \alpha\}$$

is already open and we can let $\phi_{\alpha+1}(t, x) = \alpha$ anyway.

Now the saturated analytic set $S(A)$ is included in the saturated coanalytic set C, so by the Invariant Separation Theorem (2.1) there is a saturated Borel set D with $A \subset S(A) \subset D \subset C$. Notice that if $D_t \neq \emptyset$, then D_t consists of exactly one $\sim_t$ equivalence class, since C has this property. Define the map $\phi_{\alpha+1}$ by

$$\phi_{\alpha+1}(t, x) = \begin{cases} \alpha, \text{ if } (t, x) \in D, \text{ and} \\ \phi_\alpha(t, x), \text{ otherwise.} \end{cases}$$

Since $(t, x) \in D$ implies $\phi_\alpha(t, x) > \alpha$, we have

$$\phi_{\alpha+1}(t, x) \leqq \phi_\alpha(t, x) \quad \text{for all } (t, x) \in B.$$

The map $\phi_{\alpha+1}$ is Borel measurable since both D and ϕ_α are Borel.

We next show that $\phi_{\alpha+1}$ is a representation. Certainly, $\phi_{\alpha+1}$ is invariant on $\sim_t$ equivalence classes. All we need to show is that if $x \prec_t y$, then $\phi_{\alpha+1}(t, x) < \phi_{\alpha+1}(t, y)$. Suppose $x \prec_t y$ and $y \in D_t$; then

$$\phi_{\alpha+1}(t, y) = \alpha, \quad (t, x) \notin C \quad \text{and}$$
$$\phi_{\alpha+1}(t, x) = \phi_\alpha(t, x) < \alpha.$$

Suppose $x \prec_t y$ and $y \notin D_t$; then

$$\phi_{\alpha+1}(t, y) = \phi_\alpha(t, y) > \phi_{\alpha+1}(t, x).$$

Thus $\phi_{\alpha+1}$ is an ordinal representation.

It remains to show that (1) and (2) hold for $\alpha + 1$. Given $\sigma < \alpha$, we have, for all $(t, x) \in B$:

$$(3) \quad \phi_{\alpha+1}(t, x) > \sigma \leftrightarrow \phi_\alpha(t, x) > \sigma.$$

It follows that $\{x:\phi_{\alpha+1}(t, x) > \sigma\}$ is open in B_t.

Now suppose $\phi_{\alpha+1}(t, x) > \alpha$. There are two sub-cases. First, suppose that $D_t \neq \emptyset$ and choose $y_0 \in D_t$. Then $\phi_{\alpha+1}(t, y_0) = \alpha$ and $y_0 \prec_t x$. Thus

$$x \in \{y:y_0 \prec_t y\} \subset \{y:\phi_{\alpha+1}(t, y) > \alpha\}.$$

Second, suppose that $D_t = \emptyset$; in this case, $\phi_{\alpha+1} = \phi_\alpha$. Since $A \subset D$, A_t is also empty and $x \notin A_t$. Thus by the definition of A, there is some n such that

$$x \in \left(B_t \cap \{y:d(x, y) < \frac{1}{n}\}\right) = \{y:\phi_{\alpha+1}(t, y) > \alpha\}.$$

In either case, it follows that $\{x:\phi_{\alpha+1} > \alpha\}$ is open in B_t. This establishes (1).

Given $\sigma < \beta < \alpha + 1$, it follows that $\sigma < \alpha$. Thus by (3):

$$\phi_{\alpha+1}(t, x) > \sigma \leftrightarrow \phi_\alpha(t, x) > \sigma$$
$$\leftrightarrow \sigma_\beta(t, x) > \sigma.$$

This establishes (2) and completes the proof of Case I.

(Case II: λ = limit). Suppose that ϕ_α has been constructed satisfying (1) and (2) for all $\alpha < \lambda$. Define the map $\phi_\lambda:B \to \kappa$ by

$$\phi_\lambda(t, x) = \min\{\phi_\alpha(t, x):\alpha < \lambda\}.$$

Clearly ϕ_λ is less than or equal to ϕ_α for all $\alpha < \lambda$.

Since $\{\phi_\alpha:\alpha < \lambda\}$ is a decreasing sequence of ordinal representations, it follows that ϕ_λ is a representation. For each $\sigma < \kappa$, we have

$$\phi_\lambda(t, x) > \sigma \leftrightarrow (\forall\ \alpha < \lambda)\ \phi_\alpha(t, x) > \sigma \quad \text{and}$$
$$\phi_\lambda(t, x) < \sigma \leftrightarrow (\exists\ \alpha < \lambda)\ \phi_\alpha(t, x) < \sigma.$$

It follows that ϕ_λ is Borel measurable.

For any $\sigma < \beta < \lambda$, any t and any x, it follows from the definition of ϕ_λ that if $\phi_\lambda(t, x) > \sigma$, then $\phi_\beta(t, x) > \sigma$. Now if $\phi_\beta(t, x) > \sigma$, then by (2) of the hypothesis, $\phi_\alpha(t, x) > \sigma$ for all $\beta \leqq \alpha < \lambda$ and, since the maps $\{\phi_\alpha:\alpha < \lambda\}$ are decreasing,

$$\phi_\alpha(t, x) \geqq \phi_\beta(t, x) > \sigma, \quad \text{if } \alpha < \beta.$$

So, if $\phi_\beta(t, x) > \sigma$, then $\phi_\lambda(t, x) > \sigma$. This establishes (2). In particular, if $\sigma < \lambda$, then

$$\phi_\lambda(t, x) > \sigma \leftrightarrow \phi_{\sigma+1}(t, x) > \sigma.$$

In other words,

$$\{x:\phi_\lambda\ (t, x) > \sigma\} = \{x:\phi_{\sigma+1}(t, x) > \sigma\}$$

and is open in B_t by the (1) of the induction hypothesis. This establishes (1) and completes the proof of Theorem 4.1.

THEOREM 4.2. *Let E be a Borel subset of the product $T \times X \times X$ of Polish spaces such that, for each t, E_t is a continuous well-founded preference order on B_t. Then E possesses a section-wise continuous Borel ordinal representation $\phi:B \to o(E)$ and a section-wise continuous Borel representation $f:B \to K$ of B onto a closed subset K of the real line.*

Proof. The first part is immediate from Theorem 4.1 and Theorem 3.2. As in the proof of Theorem 3.3, let $i:o(E) \to K$ be a continuous order isomorphism of $o(E)$ onto a closed subset K of the real line (given by Lemma 1.2). Let ϕ be the section-wise continuous ordinal representation as in the first part. Finally, let $f = i \circ \phi$.

It should be pointed out that the general Question (1) of [7] remains open: whether every section-wise continuous Borel preference order has a section-wise continuous Borel (or even $\mathscr{B}\mathscr{A}$-measurable) econ. 71980), 165-173.

Added in proof. Some results similar to those in [7] were obtained by A. Wieczorek, J. Math. Econ. 7 (1980), 165-173.

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