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A representation theorem for the second dual of $C[0, 1]$

by

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Abstract. Assuming the continuum hypothesis is true and the cardinality of $\operatorname{ca}(S, \Sigma)$ is $2^{\aleph_0}$, (as is the case in $C^*[0, 1]$), an integral representation of the functionals, T, of the dual of $\operatorname{ca}(S, \Sigma)$ is given: $T(\mu) = \int_S \psi d\mu$. Here, ψ is a real-valued function defined on Σ and the approximating sums are of the form $\Sigma\psi(E)\mu(E)$, where the sum is over all sets E of some partition of the space S. The integral is the limit of the approximating sums over the directed set of partitions.

Let $\mathfrak{M}[0, 1]$ denote the space of all real-valued, countably additive, regular set functions defined on the σ-algebra, $\mathfrak{B}$, of all Borel subsets of the closed interval $[0, 1]$, with the norm of a function, μ, being the total variation of μ. Let $C[0, 1]$ denote the space of all real-valued continuous functions on $[0, 1]$, with the norm of a function f being the least upper bound of $|f|$ on $[0, 1]$. The space $\mathfrak{M}[0, 1]$ is isometrically isomorphic to $C^*[0, 1]$, the first dual of $C[0, 1]$, ([2], p. 252).

Kakutani has shown that there is a compact Hausdorff space K such that $\mathfrak{M}^*[0, 1]$ is isometric and lattice isomorphic to $C(K)$ [3]. Yu Sreider has shown [7] that each functional T in $\mathfrak{M}^*[0, 1]$ can be represented as follows:

$$T(\mu) = \int_{[0,1]} f_\mu(t)\, d\mu(t),$$

where $f_\mu(t)$ is a "generalized function" meaning a function of points t in $[0, 1]$ and of measures μ in $\mathfrak{M}[0, 1]$.

A. P. Artemenko [1] proved that if $\{\mu_\alpha\}_{\alpha\in I} \subset \mathfrak{M}[0, 1]$ is a maximal set of mutually singular measures (all measures of the form "a value at a point" belong to which), then

1) For each measure $\mu \in \mathfrak{M}[0, 1]$ there exist measures $\nu_i \in \mathfrak{M}[0, 1]$, $\nu_i \ll \mu_{\alpha_i}$ such that $\mu = \sum_{i=1}^{\infty} \nu_i$.

2) For any functional $T \in \mathfrak{M}^*[0, 1]$ there exist functions $f_\alpha \in L_\infty(\mu_\alpha)$ such that

$$T\mu = \sum_{i=1}^{\infty} \int_0^1 f_{\alpha_i} d\mu \quad \text{where } \mu = \sum_{i=1}^{\infty} \nu_i \text{ and } \nu_i \ll \mu_{\alpha_i}.$$

The purpose of this paper is to show that assuming the continuum hypothesis is true, each functional T in $\mathfrak{M}^*[0,1]$ can be represented as:

$$T(\mu) = \int_{[0,1]} \psi d\mu,$$

where ψ is a bounded real-valued function defined on the Borel subsets of $[0,1]$ and the integral is the limit of approximating sums on the directed set of subdivisions or partitions on $[0,1]$.

Remark. The techniques employed here can be extended to give an integral representation of the same type of the bounded linear functions on the space $\mathrm{ca}(S, \Sigma)$ of all real-valued countably additive set functions defined on a σ-algebra, Σ, of subsets of a set S, provided that the cardinality of $\mathrm{ca}(S, \Sigma)$ is $2^{\aleph_0}$.

DEFINITIONS. "*D is a subdivision of* $[0,1]$" means that D is a finite collection of disjoint Borel sets filling up the interval $[0,1]$ and "*D' refines D*" means D' is a subdivision of $[0,1]$ and each set in D' is a subset of some set in D[ ]. If ψ and μ are real-valued functions on B, then "*the number w is the integral of ψ with respect to μ*" means that if $\varepsilon > 0$, then there is a subdivision D of $[0,1]$ such that if D' refines D, then

$$\Big|\sum_{\text{all } B \text{ in } D'} \psi(B)\mu(B) - w\Big| < \varepsilon.$$

The integral of ψ with respect to μ is denoted by $\int_0^1 \psi d\mu$. This is an integral of the Kolmogorov–Burkhill type [6]. This integral is linear in both variables.

The main result of this paper is the following theorem.

THEOREM. *Suppose $2^{\aleph_0} = \aleph_1$. Then T is a bounded linear functional on $\mathfrak{M}[0,1]$ if and only if there is a bounded, real-valued function ψ defined on $\mathfrak{B}$ such that for each μ in $\mathfrak{M}[0,1]$, ψ is μ-integrable on $[0,1]$ and*

$$T(\mu) = \int_0^1 \psi d\mu. \tag{1}$$

Remark. If a functional T on $\mathfrak{M}[0,1]$ is defined by equation (1), where ψ is a bounded real-valued function on $\mathfrak{B}$, then T is linear and it is bounded, since

$$|T(\mu)| = \Big|\int_0^1 \psi d\mu\Big| \leqslant (\operatorname*{l.u.b}_{B \text{ in } \mathfrak{B}} |\psi(B)|)\cdot\|\mu\|.$$

In order to prove the converse, let $\{\mu_\alpha\}_{\alpha\in I} \subset \mathfrak{M}[0,1]$ be a maximal set of mutually singular measures. We can assume that the measures are positive. Since card $I = 2^{\aleph_0}$ and the continuum hypothesis is assumed, the index set I can be ordered into type Ω. Let $F = \{\mu = \sum_{i=1}^{n} \nu_i \colon \nu_i \ll \mu_{\alpha_i}\}$.

Of course, F is dense in $\mathfrak{M}[0,1]$. Let $T \in \mathfrak{M}^*[0,1]$ be a non-negative functional, and $(f_\alpha)_{\alpha \in I}$ a sequence of functions defined by T (by Artemenko's characterization). Obviously, $f_\alpha \geqslant 0$.

For each γ and $\alpha, 1 \leqslant \gamma < \alpha < \Omega$, let $B_{\gamma\alpha}$ be a Borel set such that $\mu_\gamma(B_{\gamma\alpha}) = 0$ and $\mu_\alpha(B'_{\gamma\alpha}) = 0$, where $B'_{\gamma\alpha}$ denotes the complement of $B_{\gamma\alpha}$. For each $\alpha, 1 < \alpha < \Omega$, let $B_\alpha = \bigcap_{\gamma<\alpha} B_{\gamma\alpha}$; $\mu_\gamma(B_\alpha) = 0$, if $\gamma < \alpha$ and $\mu_\alpha(B'_\alpha) = 0$.

If B is a Borel set and there is some $\alpha, 1 < \alpha < \Omega$ such that $B \subseteq B_\alpha$ and $\mu_\alpha(B) > 0$, then B does not have these properties with respect to any other ordinal number $\gamma, 1 < \gamma < \Omega$ and $\mu_1(B) = 0$. It follows that the following function is well-defined for each Borel set B:

$$\psi(B) = \begin{cases} \text{g. l. b. } f_1(B), & \text{if } \mu_1(B) > 0, \\ \text{g. l. b. } f_\alpha(B), & \text{if } B \subseteq B_\alpha \text{ and } \mu_\alpha(B) > 0 \\ & \text{for some } \alpha, 1 < \alpha < \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

The function ψ is a nonnegative, real-valued function defined on B and $\psi(B) \leqslant |T|$, for each Borel set B.

Suppose ν is a nonnegative measure and $\nu \ll \mu_\alpha$, for some $\alpha, 1 < \alpha < \Omega$. Let $\varepsilon > 0$ and let D be a subdivision of $[0,1]$ which is a refinement of the subdivision $\{B_\alpha, B'_\alpha\}$ and such that if D' refines D, then

$$\varepsilon > T(\nu) - \sum_{D'} \big(\text{g. l. b. } f_\alpha(B)\big)\nu(B).$$

Suppose D' refines D. If $\nu(B) > 0$, then $\mu_\alpha(B) > 0$ and $B \subseteq B_\alpha$. Hence, $\sum_{D'} \big(\text{g. l. b. } f_\alpha(B)\big)\nu(B) = \sum_{D'} \psi(B)\nu(B)$. Thus,

$$\varepsilon > \Big| T(\nu) - \sum_{D'} \psi(B)\nu(B) \Big|.$$

Using linearity arguments, it follows that ψ is integrable for all $\mu \in F$ and using convergence arguments, ψ is integrable for all $\mu \in \mathfrak{M}[0,1]$. Let $T'(\mu) = \int_0^1 \psi d\mu$ for $\mu \in \mathfrak{M}[0,1]$. Since $T' \in M^*[0,1]$ and $T(\mu) = T'(\mu)$ for $\mu \in F$, we have $T(\mu) = T(\mu') = \int_0^1 \psi d\mu$ for $\mu \in \mathfrak{M}[0,1]$.

The general representation theorem follows from the facts that every bounded linear functional on $\mathfrak{M}[0,1]$ is the difference of two nonnegative bounded linear functionals on $\mathfrak{M}[0,1]$ [4], and that the integral is linear in the first variable.

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