MEASURABLE ONE-TO-ONE SELECTIONS AND TRANSITION KERNELS

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1. Introduction. In the present paper we continue the study of measurable one-to-one selections, started by the second author in [7], and answer most of the questions raised there.

Let X and Y be analytical topological spaces, $R \subset X \times Y$ a Borel set, and μ a probability measure on Y. A map f from a subset B of Y to X is called a selection for R if $f(y) \in R^y = \{x \in X \mid (x, y) \in R\}$ for all $y \in B$, i.e. if the graph of f is contained in R. We will show that, if R^y is uncountable for μ — a.e. $y \in Y$, then there exists a Borel subset of B of Y with $\mu(B) = 1$ and $2^{\aleph_0}$ Borel measurable one-to-one selections for R which are defined on B and have pair-wise disjoint graphs. If, in addition, X carries a probability measure λ such that the set $R_x = \{y \in Y \mid (x, y) \in R\}$ is uncountable for λ — a.e. x, then there are Borel sets $A \subset X$ and $B \subset Y$ of full λ — resp. μ — measure and a Borel isomorphism from A to B whose graph is contained in R. While there are many *almost everywhere* defined, Borel measurable one-to-one selections we show by an example that there need not be any *everywhere* defined, Borel measurable one-to-one selection, even if $X = Y = [0, 1]$, μ is Lebesgue measure and $\mu(R^y) = 1$ for all $y \in Y$. In contrast to this last result we prove that Martin's axiom implies the existence of an everywhere defined, universally measurable one-to-one selection, provided $\lambda(R^y) > 0$ for all $y \in Y$, where λ is any atomless measure on X. Counterexamples show that these results cannot be easily improved: There is a Borel set $R \subset [0, 1] \times [0, 1]$ with R^y uncountable for every $y \in Y$ such that R does not admit an everywhere defined, universally measurable one-to-one selection. There is also a Borel set $R \subset [0, 1] \times [0, 1]$ whose fibers R_x and R^y have positive Lebesgue measure but which does not have a universally measurable bijective selection.

Our methods of proof rely on the fact that, if R^y is uncountable for μ

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$-$ a.e. $y \in Y$, then there exists a transition kernel $(\sigma_y)_{y \in Y}$ from Y onto X such that σ_y is atomless and supported by R^y for μ $-$ a.e. $y \in Y$. Analyzing techniques employed by Mokobodzki [8] we deduce the following theorem on many-to-one selections from the result stated above: If R^y is uncountable for μ $-$ a.e. $y \in Y$ then, given any finite measure λ on X, there exists a λ-nullset A in X and a Borel measurable map $g: A \to Y$, whose graph is contained in R, such that $g^{-1}(y)$ is uncountable for μ $-$ a.e. $y \in Y$.

This last theorem is the key to all positive results concerning one-to-one selections proven in this paper.

2. Auxiliary results on transition kernels. Let X and Y be analytic topological spaces. Let π_X and π_Y denote the canonical projections from $X \times Y$ onto X and Y respectively. For a Borel measure σ on $X \times Y$ let $\pi_X(\sigma)$ and $\pi_Y(\sigma)$ stand for the marginal measures on X and Y. The term "measure" is always used synonymously for "Borel measure." If a measure σ' is absolutely continuous with respect to a measure σ we write $\sigma' \ll \sigma$.

A *transition kernel* $(\sigma_y)_{y \in Y}$ from Y to X is a family $(\sigma_y)_{y \in Y}$ of probability measures on X such that, for each Borel set $A \subset X$, the map $y \mapsto \sigma_y(A)$ is Borel measurable.

The following proposition was inspired by the methods used by Mokobodzki [8].

2.1. PROPOSITION. *Let X be a Polish space, Y an analytic space, μ a probability measure on Y, $(\sigma_y)_{y \in Y}$ a transition kernel from Y to X, and $\sigma = \int \sigma_y \otimes \epsilon_y \, d\mu(y)$. Suppose there is a finite measure λ on X such that, for every measure σ' on $X \times Y$ with $\pi_Y(\sigma') \leq \mu$ and $\sigma' \ll \sigma$, one has $\pi_X(\sigma') \leq \lambda$. Then σ_y is atomic for μ $-$ a.e. $y \in Y$.*

Proof. Since X is Polish, it can be checked that the set $\{(x, y) \in X \mid x \in \operatorname{supp} \sigma_y\}$ is Borel in $X \times Y$ and the map $y \mapsto \operatorname{card}(\operatorname{supp} \sigma_y)$ from Y to $[0, +\infty]$ is Borel measurable. Define $Y_\infty = \{y \in Y \mid \operatorname{card}(\operatorname{supp} \sigma_y) = \infty\}$ and assume $\mu(Y_\infty) > 0$. Using the Jankov-von Neumann measurable choice theorem repeatedly one can show that, for every $n \in IN$, there exists a Borel set $B \subset Y_\infty$ with $\mu(B) = \mu(Y_\infty)$ and Borel measurable maps $h_1, \ldots, h_n$: $B \to X$ satisfying $h_i(y) \in \operatorname{supp} \sigma_y$ for every $y \in B$ and graph $(h_i) \cap$ graph $(h_j) = \emptyset$ for $i \neq j$. Given $\epsilon > 0$, Lusin's theorem implies the existence of a compact set $K \subset B$ with $\mu(K) > \mu(B) - \epsilon$ such that $h_{i|K}$ is continuous for each $i \in \{1, \ldots, n\}$. Let d be a complete metric inducing the topology of X. Then $\delta = \min\{d(h_i(y), h_j(y)) \mid y \in K, i \neq j\}$ is strictly positive. For $r = \delta/4$ let $P_1, P_2, \ldots$ be a partition of X into Borel sets of diameter less than r and define

$$B_{i,k} = \{y \in K \mid \sigma_y(\{x \in P_k \mid d(x, h_i(y)) \leq r\}) > 0\}$$

$(i = 1, \ldots, n; k = 1, 2, \ldots)$. Since the set $E = \{(x, y) \in X \times Y | x \in P_k$ and $d(x, h_i(y)) \leq r\}$ is a Borel set it follows by standard arguments that $y \mapsto \sigma_y(E^y)$ is Borel measurable and, therefore, that $B_{i,k}$ is a Borel set in Y. The definition of the $B_{i,k}$'s implies that

$$B_{i,k} \cap B_{j,k} = \emptyset \qquad \text{for } i \neq j$$

and

$$(*) \qquad \bigcup_k B_{i,k} = K.$$

Next define
$$\sigma_y^k = \begin{cases} \dfrac{1}{\sigma_y(P_k)} \sigma_{y|P_k}, & \sigma_y(P_k) > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Then, for every Borel set $A \subset X$, the map $y \to \sigma_y^k(A)$ is Borel measurable and for $\sigma^k = \int \sigma_y^k \otimes \epsilon_y \, d\mu(y)$ we have $\pi_y(\sigma^k) \leq \mu$ and $\sigma^k \ll \sigma$. Due to our assumption concerning λ we obtain

$$\begin{aligned} \lambda(P_k) \geq \pi_X(\sigma^k)(P_k) &= \int_{\{y \in Y | \sigma_y(P_k) > 0\}} (1/\sigma_y(P_k) d\mu(y) \\ &= \mu(\{y \in Y | \sigma_y(P_k) > 0\}). \end{aligned}$$

It follows immediately from the definition and the properties of the sets $B_{i,k}$ that

$$\mu(\{y \in Y | \sigma_y(P_k) > 0\}) \geq \mu\left(\bigcup_{i=1}^n B_{i,k}\right) = \sum_{i=1}^n \mu(B_{i,k})$$

and hence that

$$\lambda(P_k) \geq \sum_{i=1}^n \mu(B_{i,k})$$

Summing both sides of this inequality with respect to k and using (*) yields

$$\lambda(X) = \sum_k \lambda(P_k) \geq \sum_{i=1}^n \sum_k \mu(B_{i,k}) \geq \sum_{i=1}^n \mu(\bigcup_k B_{i,k}) = n\mu(K).$$

Since $\mu(K) \geq \mu(Y_\infty) - \epsilon$, $n \in IN$, and $\epsilon > 0$ were arbitrary, this implies $\lambda(X) = \infty$, a contradiction. Hence we have $\mu(Y_\infty) = 0$ and, therefore, card (supp σ_y) $< \infty$ for μ − a.e. $y \in Y$. In particular σ_y is atomic for μ − a.e. $y \in Y$.

Remark. By carefully analyzing the proof of the above proposition one can obtain the following characterization of a.e. atomic transition kernels.

Under the assumptions of Proposition 2.1 the following conditions are equivalent:

(i) σ_y is atomic for μ− a.e. $y \in Y$.

(ii) There exists a sequence of measures $(\sigma_n)_{n \in IN}$ on $X \times Y$ and a sequence of measures $(\lambda_n)_{n \in IN}$ on X such that $\sigma = \sup \sigma_n$ and, for each $\sigma' \ll \sigma_n$ with $\pi_y(\sigma') \leq \mu$, one has $\pi_X(\sigma') \leq \lambda_n$.

2.2. Proposition. *Let X and Y be analytic spaces, μ a probability measure on Y, and $R \subset X \times Y$ a Borel set with R^y uncountable for μ − a.e. $y \in Y$. Then there exists a transition kernel $(\sigma_y)_{y \in Y}$ from X to Y such that σ_y is atomless and supported by R^y for μ − a.e. $y \in Y$.*

Proof. Let $M^1_+(X)$ denote the space of all probability measures on X equipped with the narrow topology ([9], p. 370). Then $M^1_+(X)$ is an analytic space ([9], p. 385). We claim that the set $D = \{(y, \nu) \in Y \times M^1_+(X) \mid \nu \text{ is atomless and } \nu \text{ is supported by } R^y\}$ is analytic. Since D is the image of $Y \times \{\nu \in M^1_+(X) \mid \nu \text{ is atomless}\}$ under the Borel measurable map $(y, \nu) \to \nu_{|R^y}$ it is enough to show that $M_a = \{\nu \in M^1_+(X) \mid \nu \text{ is atomless}\}$ is a Borel set in $M^1_+(X)$. Because the Borel field of X is countably generated ([9], p. 108), there exists a countable field $\mathfrak{A}$ generating the Borel field of X. Then a measure $\nu \in M^1_+(X)$ is atomless if and only if, for every $\epsilon > 0$, there exist $A_1, \ldots, A_n \in \mathfrak{A}$ with $A_1 \cup \ldots \cup A_n = X$ and $\nu(A_i) < \epsilon$ for $i = 1, \ldots, n$. Hence we obtain

$$\begin{aligned} M_a &= \{\nu \in M^1_+(X) \mid \forall \epsilon > 0 \, \exists A_1, \ldots, A_n \in \mathfrak{A}: A_1 \cup \ldots \cup A_n \\ &\quad = X \text{ and } \nu(A_i) < \epsilon, i = 1, \ldots, n\} \\ &= \bigcap_{m \in IN} \bigcup_{\substack{A_1, \ldots, A_n \in \mathfrak{A} \\ X = A_1 \cup \ldots \cup A_n}} \bigcap_{i=1}^{n} \left\{\nu \in M^1_+(X) \mid \nu(A_i) < \frac{1}{m}\right\} \end{aligned}$$

Thus M_a is a Borel set and our claim is proved.

Because R^y is an uncountable Borel subset of X for μ — a.e. $y \in Y$, R^y is itself an uncountable analytic space. Thus ([2], p. 118) R^y contains a Cantor space and therefore supports an atomless probability measure. So, we have $D_y \neq \emptyset$, for μ — a.e. $y \in Y$.

According to the Jankov-von Neumann measurable choice theorem, there is a Borel set $B \subset Y$ with $\mu(B) = 1$ and a Borel measurable map θ: $B \to M_1^+(X)$ with $\theta(y) \in D_y$ for every $y \in B$. Let $\nu \in M_+^1(X)$ be arbitrary and define

$$\sigma_y = \begin{cases} \theta(y), & y \in B \\ \nu, & y \in Y \backslash B. \end{cases}$$

Then $(\sigma_y)_{y \in Y}$ obviously has the desired properties.

Remark. For compact metrizable spaces X and Y the above proposition can, implicitly, be found in Mokobodzki [8]. The proof given here resembles Mokobodzki's proof. The second author of the present paper proved the result independently using the results of [1].

Our next lemma connects the results obtained so far and will be applied in section 3.

2.3. LEMMA. *Let X, Y, R and μ be as in Proposition 2.2. Then, given any finite Borel measure λ on X, there exists a Borel set $A \subset X$, a Borel set $B \subset Y$ with $\mu(B) > 0$, and a transition kernel $(\sigma_y)_{y \in B}$ from B to A with the following properties*:

(i) *For every $y \in B$ the measure σ_y is atomless and supported by $A \cap R^y$,*

(ii) *For every Borel set $Q \subset X$ one has $\lambda(A \cap Q) \leq \int_B \sigma_y(A \cap Q) d\mu(y)$.*

Proof. By Proposition 2.2 there exists a transition kernel $(\theta_y)_{y \in Y}$ from Y to X such that θ_y is atomless and supported by R^y for μ — a.e. $y \in Y$. Let $\theta = \int \theta_y \otimes \epsilon_y d\mu(y)$. We want to apply Proposition 2.1 which is not directly possible since X is not Polish. Therefore we restrict our attention to a compact set $K \subset X$ with $\theta(K \times Y) > 0$. Then the set $E = \{y \in Y \mid \theta_y(K) > 0\}$ has positive μ-measure and the transition kernel $(((1/\theta_y(K)) \cdot \theta_{y|K})_{y \in E}$ from E to K is atomless. Since K is metrizable Proposition 2.1 is applicable and yields the existence of a measure $\theta' \ll \int (1/\theta_y(K) \cdot \theta_{y|K}) \otimes \epsilon_y d\mu(y)$ with $\pi_Y(\theta') \leq \mu$ and of a Borel set $A' \subset K$ such that $\pi_X(\theta')(A') > \lambda(A')$. By the Hahn decomposition theorem there exists a Borel set A

$\subset A'$ with $\pi_X(\theta')(A) > \lambda(A)$ and $\pi_X(\theta')(A \cap Q) \geq \lambda(A \cap Q)$ for all Borel sets Q in X. Since $\theta' \ll \theta$ there exists a Borel measurable function $g: X \times Y \to IR_+$ with $\theta'(P) = \int_P g(x, y) d\theta(x, y) = \iint g(x, y) 1_p(x, y) d\theta_y(X) d\mu(y)$ for all Borel sets $P \subset X \times Y$. Since $\pi_y(\theta') \leq \mu$ we obtain $\int g(x, y)\, d\theta_y(x) \leq 1$ for μ – a.e. $y \in Y$. Now let $B = \{y \in Y \mid 0 < \int_A g(x, y) d\theta_y(x) \leq 1\}$ and define, for $y \in B$,

$$\sigma_y = \frac{1_A}{\int_A g(x, y) d\theta_y(x)} \cdot g(\circ, y)\theta_y.$$

Then $(\sigma_y)_{y \in B}$ is the transition kernel we are looking for.

3. Many-to-one selections. It is the purpose of this section to prove the following theorem.

3.1 THEOREM. *Let X and Y be analytic spaces, μ a probability measure on X, and $R \subset X \times Y$ a Borel set with R^y uncountable for μ – a.e. $y \in Y$, λ a probability measure on X. Then there exists a $\mathcal{K}_\sigma$-set $F \subset X$ with $\lambda(F) = 0$ and a Borel measurable map $g: F \to Y$ such that $g(x) \in R_x$ for all $x \in F$ and $g^{-1}(y)$* is uncountable for μ – a.e. $y \in Y$.

For the rest of this section X, Y, R, μ and λ will be as in the above theorem, whose proof will be split up into a series of lemmas and propositions.

3.2. LEMMA. *For every $\epsilon > 0$ there exists a Borel set $A \subset X$ such that $\lambda(A) \leq \epsilon$ and $R^y \cap A$ is uncountable for μ –* a.e. $y \in Y$.

Proof. Let $\mathcal{B}(X)$ (resp. $\mathcal{B}(Y)$) denote the Borel field of X (resp. Y). Consider the collection $\mathcal{H}$ of all $\mathcal{F} \subset \mathcal{B}(X) \times \mathcal{B}(Y)$ which have the following properties:

(i) If $(D, E), (D', E') \in \mathcal{F}$ and $(D, E) \neq (D', E')$, then $D \cap D' = \emptyset$ and $E \cap E' = \emptyset$.

(ii) If $(D, E) \in \mathcal{F}$ then $\mu(E) > 0$ and there exists a transition kernel $(\sigma_y)_{y \in E}$ from E to D which satisfies

a) σ_y is atomless and supported by $R^y \cap D$ for μ – a.a. $y \in E$,

b) For every Borel set $Q \subset X$ one has $\int_E \sigma_y(Q \cap D) d\mu(y) \geq 1/\epsilon\, \lambda(Q \cap D)$.

By Lemma 2.3 the collection $\mathcal{H}$ is nonempty. Inclusion is obviously an inductive partial order on $\mathcal{H}$. Let $\mathcal{F}_o$ be a maximal element in $\mathcal{H}$. Then it follows from (i) and the fact that $\mu(E) > 0$ for all $(D, E) \in \mathfrak{F}_o$ that $\mathfrak{F}_o$ is at most countable. Let $A = \bigcup\{D \mid (D, E) \in \mathfrak{F}_o\}$ and $B = \bigcup\{E \mid (D, E) \in \mathfrak{F}_o\}$. We will show that $\mu(\{y \in Y\backslash B \mid R^y \cap A \text{ at most countable}\}) = 0$. The set $\{y \in Y\backslash B \mid R^y \cap A \text{ at most countable}\}$ is coanalytic ([3], p. 496), hence μ-measurable. Assume $\mu(\{y \in Y\backslash B \mid R^y \cap A \text{ at most countable}\}) > 0$. Then there is a Borel set $E' \subset Y\backslash B$ such that $\mu(E') > 0$ and $R^y \cap (X\backslash A)$ is uncountable for all $y \in E'$. By Lemma 2.3 we can find a Borel set $E_o \subset E'$, a Borel set $D_o \subset X\backslash A$ and a transition kernel $(\sigma_y)_{y \in E_o}$ from E_o to D_o with properties a) and b) in (ii). This contradicts the maximality of $\mathfrak{F}_o$. Thus $R^y \cap A$ is uncountable for μ — a.e. $y \in Y\backslash B$. For μ — a.e. $y \in B$ the set $R^y \cap A$ carries an atomless probability measure and is, therefore, also uncountable. Hence we have proved that $R^y \cap A$ is uncountable for μ — a.e. $y \in Y$.

Now we will show that $\lambda(A) \leq \epsilon$. For $(D, E) \in \mathfrak{F}_o$ let $(\sigma_y^{D,E})_{y \in D}$ be a transition kernel as described in (ii). Then we get

$$\frac{1}{\epsilon}\lambda(A) = \sum_{(D,E)\in\mathfrak{F}_o} \frac{1}{\epsilon}\lambda(D) \leq \sum_{(D,E)\in\mathfrak{F}_o} \int_E \sigma_y^{D,E}(D)\,d\mu(y)$$

$$\leq \sum_{(D,E)\in\mathfrak{F}_o} \mu(E) \leq \mu(Y) = 1,$$

which proves $\lambda(A) \leq \epsilon$.

3.3. LEMMA. *For every $\epsilon > 0$ and every $\delta > 0$ there exist disjoint Borel sets $A_1, A_2 \subset X$ with $\lambda(A_1 \cup A_2) \leq \delta$ and $\mu(\{y \in Y \mid R^y \cap A_1$ and $R^y \cap A_2$ uncountable$\}) \geq 1 - \epsilon$.*

Proof. By Prop. 2.2 there exists a transition kernel $(\sigma_y)_{y \in Y}$ from Y to X such that σ_y is atomless and supported by R^y for μ — a.e. $y \in Y$. Applying Lemma 3.2 first to the measure $\lambda' = \int \sigma_y d\mu(y)$ and then to λ yields the existence of a Borel set $A_1 \subset X$ such that $\int \sigma_y(A_1) d\mu(y) \leq \epsilon$, $\lambda(A_1) \leq \delta/2$, and $R^y \cap A_1$ is uncountable for μ — a.e. $y \in Y$. Since $\int \sigma_y(A_1) d\mu(y) \leq \epsilon$ we obtain $\mu(\{y \in Y \mid \sigma_y(A_1) = 1\}) \leq \epsilon$ and, therefore, $\mu(\{y \in Y \mid \sigma_y(X\backslash A_1) > 0\}) \geq 1 - \epsilon$. Since σ_y is atomless for μ — a.e. $y \in Y$ this implies that $\mu(\{y \in Y \mid R^y\backslash A_1 \text{ uncountable}\}) \geq 1 - \epsilon$. By Lemma 3.2 there is a Borel set $A_2 \subset X\backslash A_1$ with $\lambda(A_2) \leq \delta/2$ and $\mu(\{y \in Y \mid R^y \cap A_2 \text{ uncountable}\}) \geq 1 - \epsilon$.

3.4. LEMMA. *For every $\epsilon > 0$ there exists a compact set $C \subset R$ with $\mu(\{y \in Y \mid C^y$ uncountable$\}) \geq 1 - \epsilon$.*

Proof. By Proposition 2.2, there is a transition kernel $(\sigma_y)_{y\in Y}$ from Y to X such that σ_y is atomless and supported by R^y for μ — a.e. $y \in Y$. Define $\sigma = \int \sigma_y \otimes \epsilon_y \, d\mu(y)$. There exists a compact set $C \subset R$ with $\sigma(R\backslash C) < \epsilon$. Since $\sigma(R\backslash C) = \int \sigma_y(R^y\backslash C^y)d\mu(y) \geq \mu(\{y \in Y | \sigma_y(C^y) = 0\})$ we have $\mu(\{y \in Y | \sigma_y(C^y) > 0\}) > 1 - \epsilon$. Because σ_y is atomless for μ — a.e. $y \in Y$ this implies the conclusion of the lemma.

3.5. LEMMA. *For every $\epsilon > 0$ there exists a compact set $K \subset X$ with $\lambda(K) = 0$, a continuous map f from K onto $\{0, 1\}^{IN}$, and a compact set $L \subset Y$ such that $\mu(Y\backslash L) < \epsilon$ and $R^y \cap f^{-1}(z) \neq \emptyset$ for all $y \in L$ and all $z \in \{0, 1\}^{IN}$.*

Proof. Let $(\epsilon_n)_{n\in IN}$ be a sequence of positive real numbers with $\Sigma_{n=1}^{\infty} \epsilon_n < \epsilon$. By Lemma 3.3, there exist two disjoint Borel sets $A(0)$, $A(1)$ in X with $\lambda(A(0) \cup A(1)) \leq \epsilon_1$ such that

$$\mu(\{y \in Y | R^y \cap A(0) \quad \text{and} \quad R^y \cap A(1) \text{ uncountable}\}) \geq 1 - \frac{\epsilon_1}{2}.$$

By Lemma 3.4. there exist compact sets $C(0) \subset (A(0) \times Y) \cap R$ and $C(1) \subset (A(1) \times Y) \cap R$ with

$$\mu(\{y \in Y | (C(0))^y \quad \text{and} \quad (C(1))^y \text{ uncountable}\}) \geq 1 - \epsilon_1.$$

By Lemma 3.3 there exist disjoint Borel sets $A(i, 0)$, $A(i, 1) \subset A(i)$ with $\lambda(A(i, 0) \cup A(i, 1)) < \epsilon_2/2$ $(i = 0, 1)$ and

$$\mu(\{y \in Y | (C(i))^y \cap A(i, 0) \text{ and}$$
$$(C(i))^y \cap A(i, 1) \text{ uncountable}, i = 0, 1\}) > 1 - \epsilon_1 - \frac{\epsilon_2}{2}.$$

Again Lemma 3.4 implies the existence of compact sets $C(i, 0) \subset C(i) \cap (A(i, 0) \times Y)$, $C(i, 1) \subset C(i) \cap (A(i, 1) \times Y)$ satisfying

$$\mu(\{y \in Y | (C(i, 0))^y \quad \text{and}$$
$$(C(i, 1))^y \text{ uncountable}, i = 0, 1\}) \geq 1 - \epsilon_1 - \epsilon_2.$$

Continuing in this way we obtain, for each $k \in IN$, pairwise disjoint Borel sets $A(i_1, \ldots, i_k)$ and compact sets $C(i_1, \ldots, i_k) \subset X \times Y((i_1, \ldots, i_k) \in \{0, 1\}^k)$ which have the following properties:

(i) $A(i_1, \ldots, i_k) \subset A(i_1, \ldots, i_{k-1})$.
(ii) $\lambda(\cup A(i_1, \ldots, i_k)) \leq \epsilon_k$.
(iii) $C(i_1, \ldots, i_k) \subset C(i_1, \ldots, i_{k-1}) \cap (A(i_1, \ldots, i_k) \times Y)$.
(iv) $\mu(\{y \in Y \mid (C(i_1, \ldots, i_k))^y$ uncountable for all $(i_1, \ldots, i_k) \in \{0, 1\}^k\}) \geq 1 - \Sigma_{\rho=1}^{k} \epsilon_\rho$.

$$\text{Define } K = \bigcap_{k \in IN} \bigcup_{(i_1, \ldots, i_k) \in \{0,1\}^k} \pi_X(C(i_1, \ldots, i_k)).$$

Then K is a compact set of λ-measure 0. The map $f: K \to \{0, 1\}^{IN}$ will be defined as follows: If $x \in K$ then there exists exactly one sequence $(i_n)_{n \in IN}$ in $\{0, 1\}^{IN}$ such that $x \in \pi_X(C(i_1, \ldots, i_k))$ for all $k \in IN$. Define $f(x) = (i_n)_{n \in IN}$. By standard arguments f is continuous and maps K onto $\{0, 1\}^{IN}$. It follows from (iv) that there exists a compact set $L \subset Y$ such that $\mu(Y \backslash L) < \epsilon$ and, for all $y \in L$, for all $k \in IN$, and for all $(i_1, \ldots, i_k) \in \{0, 1\}^k$ the set $(C(i_1, \ldots, i_k))^y$ is uncountable, hence not empty. Now let $z = (i_n)_{n \in IN} \in \{0, 1\}^{IN}$ be fixed. Since $(C(i_1, \ldots, i_k))^y \subset \pi_X(C(i_1, \ldots, i_k)) \cap (C(i_1))^y$ and these last sets are compact we obtain

$$f^{-1}(z) \cap R^y \supset f^{-1}(z) \cap (C(i_1))^y = \bigcap_{k \in IN} \pi_X(C(i_1, \ldots, i_k)) \cap (C(i_1))^y \neq \emptyset$$

for every $y \in L$. This completes the proof of the lemma.

Lemma 3.5 allows us to prove the following version of a result of Mokobodzki [8]:

3.6. THEOREM. *Let X and Y be analytic spaces, μ a probability measure on Y, and $R \subset X \times Y$ a Borel set. Then the following properties of R are equivalent:*

(i) *R^y is at most countable for μ − a.e. $y \in Y$.*
(ii) *Every family $(A_i)_{i \in I}$ of pairwise disjoint Borel subsets of X, satisfying $\mu(\{y \in Y \mid R^y \cap A_i \neq \emptyset\}) > 0$ for every $i \in I$, is at most countable.*
(iii) *There exists a finite measure λ on X such that, for every Borel set $A \subset X$, $\lambda(A) = 0$ implies $\mu(\{y \in Y \mid R^y \cap A \neq \emptyset\}) = 0$.*

Proof. (i) ⇒ (ii) and (i) ⇒ (iii): Assuming (i) one can find a sequence $(f_n)_{n \in IN}$ of μ-measurable functions from Y to X such that $R^y \subset \{f_n(y) \mid n \in$

$IN\}$ for μ − a.e. $y \in Y$ (For sets R in IR^n this was proved by Lusin [4], p. 244. But the result can easily be generalized to the present situation). To prove (iii) let $(A_i)_{i \in I}$ be a family of pairwise disjoint Borel sets in X satisfying $\mu(\{y \in Y | R^y \cap A_i \neq \emptyset\}) > 0$ for every $i \in I$. Then, for each $i \in I$, there exists an $n_i \in IN$ with $\mu(f_{n_i}^{-1}(A)) > 0$. Since, for each $n \in IN$, the family $(f_n^{-1}(A_i))_{i \in I}$ consists of pairwise disjoint sets, we know that only countably many of these sets have positive μ-measure. Thus $I_n = \{i \in I | n_i = n\}$ is at most countable. Because $I = \bigcup_{n \in IN} I_n$ this implies that I is at most countable.

To prove (iii) define $\lambda = \Sigma_{n=1}^{\infty} 1/2^n \mu \circ f_n^{-1}$. Then, for any Borel set $A \subset X$, $\lambda(A) = 0$ implies $\mu(f_n^{-1}(A)) = 0$ for all $n \in IN$ and hence $\mu(\bigcup_{n \in IN} f_n^{-1}(A)) = 0$. Since $\{y \in Y | R^y \cap A \neq \emptyset\}$ is contained in $\bigcup_{n \in IN} f_n^{-1}(A)$ this proves (iii).

(iii) ⇒ (ii): This implication follows immediately from the fact that for a finite measure, a family of pairwise disjoint sets of positive measure is at most countable.

(ii) ⇒ (i): If $\mu(\{y \in Y | R^y \text{ uncountable}\}) > 0$ then, by Lemma 3.5, there exists a family $(A_z)_{z \in \{0,1\}^{IN}}$ of pairwise disjoint Borel sets in X with $\mu(\{y \in Y | R^y \cap A_z \neq \emptyset\}) > 0$ for all $z \in \{0, 1\}^{IN}$.

3.7. Proposition. *Let X and Y be analytic spaces, μ a probability measure on Y, λ a finite measure on X, and $R \subset X \times Y$ a Borel set such that R^y is uncountable for μ − a.e. $y \in Y$. Then, given $\epsilon > 0$, there exists a compact set $M \subset X$ with $\lambda(M) = 0$ and a continuous map $g: M \to Y$ such that $g(x) \in R_x$ for all $x \in M$ and $\mu(\{y \in Y | g^{-1}(y) \text{ uncountable}\}) > 1 - \epsilon$.*

Proof. We claim that there is an analytic set $A \subset X$ with $\lambda(A) = 0$ and a Borel measurable map $h: A \to Y$ satisfying $h(x) \in R_x$ for all $x \in A$ and $\mu(\{y \in Y | h^{-1}(y) \text{ uncountable}\}) \geq 1 - \epsilon/2$.

Assume for the moment that the claim has been proved. Then, by Lemma 3.4, there exists a compact set $C \subset$ graph (h) such that $\mu(\{y \in Y | C^y \text{ uncountable}\}) > 1 - \epsilon$. Define $M = \pi_X(C)$ and $g = h_{|M}$. Then M and g have the properties required in the proposition. Thus it remains to prove our claim. To this end let L, K and f be as in Lemma 3.5.

Case 1. L is countable. Let $(A_y)_{y \in L}$ be a partition of $\{0, 1\}^{IN}$ into uncountable Borel sets. Define $A = \bigcup_{y \in L} (f^{-1}(A_y) \cap R^y)$ and $h: A \to L$ by $h(x) = y$ if $x \in f^{-1}(A_y) \cap R^y$. Then A is a Borel set and h is a well-defined Borel measurable map with $h(x) \in R_x$ for all $x \in A$. Since $f^{-1}(z) \cap R^y \neq \emptyset$ for all $z \in \{0, 1\}^{IN}$ we deduce that $h^{-1}(y)$ is uncountable for all $y \in L$.

Case 2. L is uncountable. Then there exists a Borel isomorphism ϕ from $\{0, 1\}^{IN}$ onto L ([3], p. 451). Let ψ be a continuous map from $\{0, 1\}^{IN}$ onto itself such that $\psi^{-1}(z)$ is uncountable for all $z \in \{0, 1\}^{IN}$. Define h from a subset of X to Y by graph $(h) =$ graph $(\phi \circ \psi \circ f) \cap R$. Since graph (h) is a Borel subset of $X \times Y$ the domain A of h is analytic and $h: A \to Y$ is Borel measurable. Since A is obviously contained in the domain of f, which is K, we obtain $\lambda(A) = 0$. For $y \in L$ we have $h^{-1}(y) = f^{-1}\psi^{-1}\phi^{-1}(y) \cap R^y$. Since $\psi^{-1}\phi^{-1}(y)$ is uncountable and $f^{-1}(z) \cap R^y \neq \emptyset$ for all $z \in \psi^{-1}\phi^{-1}(y)$ we deduce that $h^{-1}(y)$ is uncountable and the proof is completed.

3.8. Corollary. *Let X and Y be analytic spaces, μ a probability measure on Y, λ a finite measure on X, and $f: X \to Y$ a Borel measurable map such that $f^{-1}(y)$ is uncountable for μ — a.e. $y \in Y$. Then, for every $\epsilon > 0$, there exists a compact set $M \subset X$ with $\lambda(M) = 0$ such that $f_{|M}$ is continuous and $\mu(\{y \in Y | f^{-1}(y) \cap M$ uncountable$\}) > 1 - \epsilon$.*

We are now prepared to give the proof of Theorem 3.1.

Proof of Theorem 3.1. According to Proposition 3.7 there exists, for each $n \in IN$, a compact set $M_n \subset X$ with $\lambda(M_n) = 0$ and a continuous map $g_n: M_n \to Y$ such that $g_n(x) \in R_x$ for all $x \in M_n$ and $\mu(\{y \in Y | g_n^{-1}(y)$ uncountable$\}) > 1 - 1/n$. By Proposition 2.2 there exists a transition kernel $(\sigma_y^n)_{y \in g_n(M_n)}$ from $g_n(M_n)$ to M_n such that σ_y^n is atomless and supported by $g_n^{-1}(y)$ for μ — a.e. $y \in g_n(M_n)$. Define

$$\sigma^n = \int_{g_n(M_n)} \sigma_y^n d\mu(y)$$

and

$$\sigma = \sum_{n=1}^{\infty} \frac{1}{2^n} \sigma^n.$$

According to Corollary 3.8 there is a compact set $K_1 \subset M_1$ with $\sigma(K_1) = 0$ and $\mu(\{y \in Y | g_1^{-1}(y) \cap K_1$ uncountable$\}) > 0$. Since $\sigma(K_1) = 0$ we have $\sigma_y^n(K_1) = 0$ for μ — a.e. $y \in g_n(M_n)$ and for every $n \in IN$. Hence $g_n^{-1}(y) \backslash K_1$ is uncountable for μ — a.e. $y \in g_n(M_n)$. By Corrollary 3.8 there is a compact set $K_2 \subset M_2 \backslash K_1$ with $\sigma(K_2) = 0$ and $\mu(\{y \in Y | g_2^{-1}(y) \cap K_2$ uncountable$\}) > 1 - 1/2$.

Since $\sigma(K_1 \cup K_2) = 0$ it follows as above that $g_n^{-1}(y) \backslash (K_1 \cup K_2)$ is uncountable for μ — a.e. $y \in g_n(M_n)$. Continuing in this manner yields a sequence $(K_n)_{n \in IN}$ of pairwise disjoint compact sets in X with $K_n \subset M_n$ and

$$\mu(\{y \in Y \mid g_n^{-1}(y) \cap K_n \text{ uncountable}\}) > 1 - \frac{1}{n}.$$

Now define $F = \bigcup_{n \in IN} K_n$ and $g: F \to Y$ by $g(y) = g_n(y)$ if $y \in K_n$. Then F is a $\mathcal{K}_\sigma$-set with $\lambda(F) = 0$ and g is Borel measurable and satisfies

$$\{y \in Y \mid g^{-1}(y) \text{ uncountable}\} \supset \bigcup_{n \in IN} \{y \in Y \mid g_n^{-1}(y) \cap K_n \text{ uncountable}\}.$$

Therefore $\mu(\{y \in Y \mid g^{-1}(y) \text{ uncountable}\}) = 1$ and the proof is completed.

In view of Proposition 3.7 one may ask whether the set F in Theorem 3.1 can be chosen compact. That, in general, this cannot be done is shown by the following example.

3.9. *Example.* Let $X = Y = [0, 1]$ and let μ and λ be Lebesgue measure. The non-empty perfect sets of $[0, 1]$ form a dense G_δ-set $\mathcal{P}$ in the space $\mathcal{K}^*([0, 1])$ of all non-empty compact subsets of $[0, 1]$ with the Hausdorff metric. Hence there exists a continuous bijection ϕ from the space $\mathcal{J}$ of all irrationals in $[0, 1]$ onto $\mathcal{P}$. Define $R \subset X \times Y$ to be equal to $\bigcup\{\phi(y) \times \{y\} \mid y \in \mathcal{J}\}$. Then R is a Borel set with R^y uncountable for μ — a.e. $y \in Y$. Now let us assume that there is a compact subset K of X with $\lambda(K) = 0$ and a Borel measurable map $g: K \to Y$ such that $g(x) \in R_x$ for all $x \in K$ and $\mu(Y \backslash g(K)) = 0$. Then we have $g^{-1}(y) \subset \phi(y) \cap K$ for μ — a.e. $y \in Y$. Since the set $\{P \in \mathcal{P} \mid P \cap K = \emptyset\}$ is non-empty and open in $\mathcal{P}$ the same is true for the set $\{y \in \mathcal{J} \mid \phi(y) \cap K = \emptyset\}$ relative to $\mathcal{J}$. Thus, this set has positive μ-measure and is contained in $Y \backslash g(K)$, a contradiction.

Theorem 3.1 also suggests the following question: Can one choose the map g in that theorem to be onto, if one assumes that R^y is uncountable for every $y \in Y$? That the answer is no, in general, can be seen by the following example.

3.10. *Example.* Let $X = Y = [0, 1]$ and let λ denote Lebesgue measure on $[0, 1]$. Then there exists a Borel set $R \subset X \times Y$ with $\lambda(R^y) = 1$ for every $y \in Y$ such that there is *no* λ-measurable set $A \subset X$ and *no* λ-measurable map g from A onto Y which satisfies $g(x) \in R_x$ for all $x \in A$.

For the proof of this statement we will need the following lemma.

3.11. LEMMA. *For* $0 < \alpha < 1$ *let* $\mathcal{K}_\alpha$ *be equal to* $\{K \subset [0, 1] \mid K$ *compact and* $\lambda(K) \geq \alpha\}$ *considered as a closed subset of* $\mathcal{K}^*([0, 1])$. *Then, for every* λ*-measurable set* $A \subset [0, 1]$, *every* λ*-measurable map* $g: A \to \mathcal{K}_\alpha$ *which satisfies* $x \in g(x)$ *for every* $x \in A$ *is not surjective.*

Proof. Let $A \subset [0, 1]$ be λ-measurable, $g: A \to \mathcal{K}_\alpha$ a λ-measurable map with $x \in g(x)$ for every $x \in A$. Assume g is onto. For $\alpha < \beta < 1$, define $\mathcal{L}_\beta = \{K \in \mathcal{K}_\alpha \mid \lambda(K) = \beta\}$. Then $\mathcal{L}_\beta$ is a Borel set in $\mathcal{K}_\alpha$. We claim that $\lambda(g^{-1}(\mathcal{L}_\beta)) \geq 1 - \beta$. If $\lambda(g^{-1}(\mathcal{L}_\beta))$ were less than $1 - \beta$, then we could find a compact set $K \subset [0, 1] \backslash g^{-1}(\mathcal{L}_\beta)$ with $\lambda(K) = \beta$; i.e. $K \in \mathcal{L}_\beta$. Since g is assumed to be onto, there would be a point $x \in A$ with $g(x) = K$, but because of $x \in g(x) \subset K \subset [0, 1] \backslash g^{-1}(\mathcal{L}_\beta)$ this is impossible. The family $(\mathcal{L}_\beta)_{\alpha < \beta < 1}$ is an uncountable family of pairwise disjoint Borel sets and, therefore, $(g^{-1}(\mathcal{L}_\beta))_{\alpha < \beta < 1}$ is an uncountable family of pairwise disjoint λ-measurable sets of positive λ-measure, a contradiction.

Proof for Example 3.10. For $n \in IN$, let $\mathcal{K}_n = \mathcal{K}_{1 - 1/(n+1)}$. Then $\pi_{n=1}^{\infty} \mathcal{K}_n$ is an uncountable compact metrizable space. Hence there exists a Borel isomorphism Φ from $[0, 1]$ onto $\pi_{n=1}^{\infty} \mathcal{K}_n$ ([3], p. 451). Let Φ_n denote the n-th component of this isomorphism. Define $R_n = \{(x, y) \in X \times Y \mid x \in \Phi_n(y)\}$ and $R = \bigcup_{n=1}^{\infty} R_n$. We claim that each of the sets R_n and, therefore, R is Borel. The set $E_n = \{(x, y, F) \in X \times Y \times \mathcal{K}^*(X) \mid \Phi_n(y) = F \text{ and } x \in F\}$ is a Borel set in $X \times Y \times \mathcal{K}^*(X)$, because it is the graph of the Borel measurable map $(x, y) \to \Phi_n(y)$ intersected with the closed set $\{(x, y, F) \in X \times Y \times K^*(X) \mid XGF\}$. Since R_n is the image of E_n with respect to the canonical projection from $X \times Y \times \mathcal{K}^*(X)$ onto $X \times Y$ which, restricted to E_n, is one-to-one, it follows that R_n is a Borel set. For each $y \in [0, 1]$ and $n \in IN$ we have $\lambda(R^y) \geq \lambda(\Phi_n(y)) \geq 1 - 1/(n + 1)$ and therefore, $\lambda(R^y) = 1$.

Now assume that $A \subset X$ is λ-measurable and g is a λ-measurable map from A onto Y satisfying $g(\mathrm{x}) \in R_x$ for every $x \in A$. Let $A_n = \{x \in X \mid (x, g(x)) \in R_n\}$. Then A_n is a λ-measurable set. The map $\Phi_n \circ g_{|A_n}: A_n \to \mathcal{K}_n$ is λ-measurable and satisifies $x \in \Phi_n(g(x))$ for every $x \in A_n$. It follows from Lemma 3.11 that $\Phi_n \circ g_{|A_n}$ is not onto. Hence there exists a $K_n \in \mathcal{K}_n$ with $K_n \notin \Phi_n \circ g(A_n)$. Let $y \in [0, 1]$ be such that $\Phi(y) = (K_n)_{n \in IN}$. Since g is assumed to be onto there is an $x \in A$ with $g(x) = y$. Because $(x, g(x)) \in R$ this implies $(x, g(x)) = (x, y) \in R_n$ for some n and, therefore, $x \in \Phi_n(y) = K_n$; i.e. $x \in A_n$ and $\Phi_n(\mathrm{g}(x)) = K_n$, a contradiction.

Remark. It should be noted that Example 3.10 and Lemma 3.11 remain true if we replace Lebesgue measure by any atomless measure.

4. Borel measurable one-to-one selections. In this section we deal with existence problems for Borel measurable one-to-one selections. Our main tool for proving positive results is Theorem 3.1. We also give some counterexamples which show that our results are, in some sense, the best possible.

Our first theorem improves Theorem 5 of [7].

4.1 THEOREM. *Let X and Y be analytic spaces, μ a probability measure on Y, λ a probability measure on X, and $R \subset X \times Y$ a Borel set with R^y uncountable for μ — a.e. $y \in Y$. Then there exists a Borel set $B \subset Y$ with $\mu(B) = 1$ and a one-to-one Borel measurable map $f: B \to X$ such that $\lambda(f(B)) = 0$ and $f(y) \in R^y$ for every $y \in B$.*

Proof. Let F and g be as in Theorem 3.1. By the Jankov-von Neumann measurable choice theorem there exists a Borel set $B \subset Y$ with $\mu(B) = 1$ and a Borel measurable map $f: B \to F$ with $g \circ f = id_B$. Then f is obviously injective.

The following is a refined version of the preceding theorem using the parametrization result of [6].

4.2 THEOREM. *Let X and Y be analytic spaces, μ a probability measure on Y, λ a finite measure on X, and $R \subset X \times Y$ a Borel set with R^y uncountable for μ — a.e. $y \in Y$. Then there exists a Borel set $B \subset Y$ with $\mu(B) = 1$ and a family $(f_i)_{i \in I}$ of power $2^{\aleph_0}$ such that*

(i) *$f_i: B \to X$ is Borel measurable and one-to-one;*
(ii) *$f_i(y) \in R^y$ for all $y \in B$;*
(iii) *$Gr(f_i) \cap Gr(f_j) = \emptyset$ if $i \neq j$;*
(iv) *$\lambda(\bigcup_{i \in I} f_i(B)) = 0$.*

Proof. According to Theorem 3.1 there exists a $\mathcal{K}_\sigma$-set $F \subset X$ with $\lambda(F) = 0$ and a Borel measurable map $g: F \to Y$ such that $g(X) \in R_x$ for all $x \in F$ and $g^{-1}(y)$ is uncountable for μ — a.e. $y \in Y$. Let $(K_n)_{n \in IN}$ be an increasing sequence of compact sets with $F = \bigcup_{n \in IN} K_n$. It follows from Proposition 2.2 that there exists a transition kernel $(\sigma_y)_{y \in Y}$ from Y to X such that σ_y is atomless and supported by $g^{-1}(y)$ for μ — a.e. $y \in Y$. Using standard arguments we can find a sequence $(L_n)_{n \in IN}$ of compact subsets of Y with the following properties:

a) $\mu(\bigcup_{n=1}^{\infty} L_n) = 1$

b) For every $y \in L_n$ the measure σ_y is atomless, supported by $g^{-1}(y)$, and satisfies $\sigma_y(K_n) > 0$.

Condition b) implies that $g^{-1}(y) \cap K_n$ is uncountable for every $y \in L_n$. Let I be a set of power $2^{\aleph_0}$. Since K_n and L_n are metrizable and compact, Theorem 2.4 and 2.5 in [6] imply that there exists a family $(f_{i,n})_{i\in I}$ of Borel measurable maps $f_{i,n}: L_n \to K_n$ with $f_{i,n}(y) \in g^{-1}(y)$ for every $y \in L_n$ and $gr(f_{i,n}) \cap gr(f_{j,n}) = \emptyset$ if $i \neq j$. Now define f_i: $\cup_{n\in IN} L_n \to X$ by $f_i(y) = f_{i,n}(y)$ if $y \in L_n \setminus \cup_{k=1}^{n-1} L_k$. Then f_i is Borel meaurable, $f_i(y) \in g^{-1}(y)$ for every $y \in \cup_{n\in IN} L_n$, and $Gr(f_i) \cap Gr(f_j) = \emptyset$ if $i \neq j$. Since $g \circ f_i = id_{\cup L_n}$ we know that f_i is one-to-one and the theorem is proved.

Remark. It has been noted in [7] (Example, p. 828) that there is a Borel set $R \subset [0, 1] \times [0, 1]$ with R^y and R_x uncountable for all $x, y \in [0, 1]$ which does not admit a Borel measurable selection defined everywhere. On the other hand it is known that R admits $2^{\aleph_0}$ everywhere defined Borel measurable selections with disjoint graphs provided there exists an atomless measure ν on Y with $\nu(R^y) > 0$ for every y ([6], p. 229).

The question, whether under this condition there exists an everywhere defined Borel measurable one-to-one selection, has a negative answer, as can be seen from the following example.

4.3. *Example.* Let $X = Y = [0, 1]$ and let λ be Lebesgue measure on $[0, 1]$. Then there exists a Borel set $R \subset X \times Y$ with $\lambda(R^y) = 1$ for all $y \in Y$ such that there is *no* one-to-one Borel measurable map f: $Y \to X$ with $f(y) \in R^y$ for all $y \in Y$.

Proof. Let R be the set defined in Example 3.10. Let us assume that f: $Y \to X$ has the properties listed above. Then $A = f(Y)$ is a Borel set and $g = f^{-1}$ is a Borel measurable map from A onto Y with $g(x) \in R_x$ for every $x \in A$. But according to Example 3.10 such a map does not exist.

Remark. The preceding example shows that it is almost hopeless to look for measure theoretic conditions on the fibers R^y of R which would ensure that there is an everywhere defined one-to-one Borel measurable selection for R. But there is a positive result of D. Maharam & A. Stone [10] which can be rephrased as follows:

Let $X = Y = [0, 1]$ and let $R \subset X \times Y$ be a Borel set, such that R^y has non-empty interior for every $y \in Y$. Then there exists a Borel measurable one-to-one map f: $Y \to X$ with $f(y) \in R^y$ for every $y \in Y$.

Our next result provides a positive answer to Question 3 in [7]. Our method of proof involves the same technique used in [7] to prove this theorem assuming almost all R^y and R_x have positive measure.

4.4. THEOREM. *Let X and Y be analytic spaces, λ a probability measure on X, μ a probability measure on Y, and $R \subset X \times Y$ a Borel set such that R_x is uncountable for λ — a.e. $x \in X$ and R^y is uncountable for μ — a.e. $y \in Y$. Then there exists a Borel set $A \subset X$ with $\lambda(A) = 1$, a Borel set $B \subset Y$ with $\mu(B) = 1$, and a Borel isomorphism f from A onto B whose graph is contained in R.*

Proof. According to Theorem 3.1 there exists a $\mathcal{K}_\sigma$-set $F \subset X$ with $\lambda(F) = 0$ and a Borel measurable map $g: F \to Y$ such that $g(x) \in R_x$ for every $x \in F$ and $g^{-1}(y)$ is uncountable for μ — a.e. $y \in Y$. Again by Theorem 3.1 there is a $\mathcal{K}_\sigma$-set $G \subset Y$ with $\mu(G) = 0$ and a Borel measurable map $h: G \to X\backslash F$ such that $h(y) \in R^y$ for every $y \in G$ and $h^{-1}(x)$ is uncountable for λ — a.e. $x \in X$. It follows from the Jankov-von Neumann measurable choice theorem combined with Lusin's theorem that there exists a $\mathcal{K}_\sigma$-set $B_1 \subset g(F)$ with $\mu(B_1) = 1$ and a Borel measurable map f_1: $B_1 \to F$ with $g \circ f_1 = id_{B_1}$ as well as a $\mathcal{K}_\sigma$-set $A_1 \subset h(G) \subset X\backslash F$ with $\lambda(A_1) = 1$ and a Borel measurable map $g_1: A_1 \to G$ with $h \circ g_1 = id_{A_1}$. Then f_1 and g_1 are one-to-one and map Borel sets onto Borel sets. In particular $A_2 = f_1(B_1)$ is a Borel set contained in F. Now define $A = A_1 \cup A_2$ and $f: A \to Y$ by

$$f(x) = \begin{cases} g_1(x), & x \in A_1 \\ f_1^{-1}(x), & x \in A_2. \end{cases}$$

Then f is a well-defined one-to-one Borel measurable map which maps Borel sets onto Borel sets. Thus, if $B = f(A)$, then f is a Borel isomorphism from A onto B. By definition we have $\lambda(A) = 1$. Since $B = f(A) \supset f_1^{-1}(A_2) = B_1$ we also deduce $\mu(B) = 1$. For $x \in A_1$ we obtain $f(x) = g_1(x)$ and hence $(x, f(x)) = (h \circ f(x), f(x)) \in R$. For $x \in A_2$ we get $f(x) = f_1^{-1}(x) = g(x) \in R_x$, hence again $(x, f(x)) \in R$.

Remark. Let $0 < \alpha < 1$. By modifying Example 4.3 one can obtain a Borel set $R \subset [0, 1] \times [0, 1]$, such that, for all $x, y \in [0, 1]$, the Lebesgue measure of R_x and R^y is greater or equal to α and R does not contain the graph of any Borel isomorphism from [0, 1] into [0, 1] (see Example 5.4). The question whether this is still true for $\alpha = 1$ remains open.

5. Universally measurable one-to-one selections. In the last section we have seen that everywhere defined Borel measurable one-to-one selections need not exist even if the fibers of the Borel set R have large measure. In contrast to this negative result we will now show (assuming Martin's axiom) that everywhere defined universally measurable one-to-one selections exist provided all fibers have positive measure (for some atomless measure).

5.1. THEOREM (MA). *Let X and Y be analytic spaces, λ an atomless probability measure on X, and $R \subset X \times Y$ a Borel set with $\lambda(R^y) > 0$ for every $y \in Y$. Then there exists a universally measurable one-to-one map $f\colon Y \to X$ with $f(y) \in R^y$ for every $y \in Y$.*

Proof. If Y is countable then the result is easily proved by induction. Therefore, suppose Y is uncountable. Let $\mathfrak{C}$ denote the smallest ordinal with the cardinality of the continuum and let $M_+^1(Y)$ stand for the space of all probability measures on Y. Let $(\mu_\alpha)_{\alpha < \mathfrak{C}}$ be an enumeration of $M_+^1(Y)$. Using Theorem 4.1, transfinite induction, the fact that under the assumption of Martin's axiom the union of less than $\mathfrak{C}$ λ-nullsets is again a λ-nullset ([5], p. 169) and $\lambda(R^y) > 0$ we define a family $(B_\alpha, f_\alpha)_{\alpha < \mathfrak{C}}$ with the following properties:

(i) B_α is Borel in Y with $\mu_\alpha(B_\alpha) = 1$;

(ii) $f_\alpha\colon B_\alpha \to X$ is one-to-one Borel measurable and satisfies $\lambda(f_\alpha(B_\alpha)) = 0$ and $f_\alpha(y) \in R^y$ for every $y \in B_\alpha$;

(iii) For $\alpha, \beta < \mathfrak{C}$ and $\alpha \neq \beta$ one has $f_\alpha(B_\alpha) \cap f_\beta(B_\beta) = \emptyset$.

We claim that $Y = \cup_{\alpha < \mathfrak{C}} B_\alpha$. Let $y_o \in Y$ be arbitrary. Then there exists an $\alpha_o < \mathfrak{C}$ with $\epsilon_{y_o} = \mu_{\alpha_o}$. Since y_o is contained in every set of full μ_{α_o}-measure we have $y_o \in B_{\alpha_o}$ and hence our claim is proved. Define $D_\alpha = B_\alpha \setminus \cup_{\beta < \alpha} B_\beta$. Then D_α is universally measurable because, under the assumption of Martin's axiom, the union of less than $\mathfrak{C}$ universally measurable sets is universally measurable ([6], p. 169). Define $f\colon Y \to X$ by $f(y) = f_\alpha(y)$ if $y \in D_\alpha$. We will show that f is universally measurable. Let $\mu \in M_+^1(Y)$ be arbitrary. Let $\Gamma = \{\gamma < \mathfrak{C} \mid \mu(D_\gamma) > 0\}$. We claim that μ is supported by $\cup_{\gamma \in \Gamma} D_\gamma$. There is an $\alpha < \mathfrak{C}$ with $\mu = \mu_\alpha$ and μ_α is supported by B_α, hence by $\cup_{\beta \leq \alpha} D_\beta$. Since Martin's axiom implies that $\cup\{D_\gamma \mid \gamma \leq \alpha \text{ and } \gamma \notin \Gamma\}$ is a μ_α-nullset our claim is proved. Because Γ is at most countable and each of the f_γ is Borel measurable, it follows that $f_{|\cup\{D_\gamma|\gamma \in \Gamma\}}$ and, therefore, f is μ-measurable. Since $\mu \in M_+^1(Y)$ was arbitrary, this proves f to be universally measurable.

The following example shows that, in Theorem 5.1, the assumption on the fibers R^y, to have positive measure, is essential.

5.2. *Example.* Let $X = Y = [0, 1]$. There exists a Borel set $R \subset X \times Y$ with R^y uncountable for every $y \in Y$ such that, for every atomless probability measure μ on Y, there is *no* μ-measurable one-to-one map $f: Y \to X$ with $f(y) \in R^y$ for every $y \in Y$.

Proof. Let $\mathcal{P}$ be the space of all non-empty perfect subsets of $[0, 1]$ (see Example 3.9). Then there exists a Borel isomorphism ϕ from $[0, 1]$ onto $\mathcal{P}$ ([3], p. 451). Define $R \subset X \times Y$ by $R = \{(x, y) \in X \times Y | x \in \phi(y)\}$. Then R is a Borel set. Let μ be an atomless probability measure on Y. Assume $f: Y \to X$ is a one-to-one μ-measurable map with $f(y) \in R^y$ for all $y \in Y$. Then there exists a Borel set $B \subset Y$ with $\mu(B) = 1$ such that $f_{|B}$ is Borel measurable. Since $f_{|B}$ is a Borel isomorphism from B onto the Borel set $f(B)$, this last set carries an atomless probability measure and contains, therefore, a non-empty perfect set K_o. For every $K \in \mathcal{P}$ and $K \subset K_o$ we obtain $f(\phi^{-1}(K)) \in K \subset f(B)$. Since f is one-to-one this implies $\phi^{-1}(K) \in B$. Let ν be an atomless probability measure on K_0. Let $\mathcal{P}_\nu = \{K \in \mathcal{P} | K \subset K_o \text{ and } \nu(K) > 0\}$. Then $g = f \circ \phi^{-1}{}_{|\mathcal{P}_\nu}$ is a Borel measurable map from $\mathcal{P}_\nu$ to K_o with $g(K) \in K$ for every $K \in \mathcal{P}_\nu$. Using the methods employed in the proof of Lemma 3.11 one can see that such a map does not exist.

In view of Theorem 4.1 one may ask whether the selection in Theorem 5.1 can be chosen to satisfy $\lambda(f(X)) = 0$. The following example shows that, in general, this is impossible.

5.3. *Example.* Let $X = Y = [0, 1]$. There exists a Borel set $R \subset X \times Y$ with R^y uncountable for every $y \in Y$ such that, for any atomless probability measure λ on X and any map $f: Y \to X$ satisfying $f(y) \in R^y$ for all $y \in Y$, the set $f(Y)$ is *not* a λ-nullset.

Proof. Let R be as in the preceding example, λ be an atomless probability measure on X, and let $f: Y \to X$ satisfy $f(y) \in R^y$ for all $y \in Y$. Assume $\lambda(f(Y)) = 0$. Then there exists a non-empty perfect set $K \subset X \backslash f(Y)$. Thus there is a $y \in Y$ with $K = R^y$ and, therefore, $f(y) \in K \subset X \backslash f(Y)$, a contradiction.

Theorem 5.1, together with Theorem 4.4, suggests the possibility that a Borel subset R of $[0, 1] \times [0, 1]$ whose fibers R_x and R^y have large Lebesgue measure for all $x, y \in [0, 1]$, might always contain the graph of a universally measurable bijection f from $[0, 1]$ onto itself. That this is not the case can be seen by the following example, which also answers Question 1 in [7] in the negative.

5.4. *Example.* Let $X = Y = [0, 1]$ and $0 < \alpha < 1$. Let λ be Lebesgue measure on [0, 1]. Then there exists a Borel set $R \subset X \times Y$ satisfying $\lambda(R_x) \geq \alpha$ and $\lambda(R^y) \geq \alpha$ for all $x, y \in Y$ such that there is *no* universally measurable map f from Y onto X with $f(y) \in R^y$ for every $y \in Y$.

Proof. Let $\mathcal{K}_\alpha$ be as in Lemma 3.11. Then there exists a Borel isomorphism ϕ from $[\alpha, 1]$ onto $\mathcal{K}_\alpha$. Define

$$R = [0, \alpha] \times [0, 1] \cup \{(x, y) \in [\alpha, 1] \times [0, 1] \,|\, y \in \phi(x)\}.$$

Then R is a Borel set with $\lambda(R^y) \geq \alpha$ and $\lambda(R_x) \geq \alpha$ for all $x, y \in [0, 1]$. Assume f is a universally measurable map from Y onto X satisfying $f(y) \in R^y$ for every $y \in Y$. Then $B = f^{-1}([\alpha, 1])$ is universally measurable. For every $y \in B$ we have $f(y) \in R^y \cap [\alpha, 1]$ and hence $y \in \phi(f(y))$. Thus $\phi \circ f_{|B}$ is a universally measurable map from B onto $\mathcal{K}_\alpha$ which satisfies $y \in f \circ \phi(y)$ for every $y \in B$. But according to Lemma 3.11, such a map does not exist.

Remark. It remains open whether an example like the above exists for $\alpha = 1$.

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