RIGOROUS MULTIFRACTAL ANALYSIS

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INTRODUCTION

The circumstance of having a fractal K, together with a probability measure ρ on the fractal, allows us to think about a "multi-fractal", where, for example, we make use of that measure to specify a dimension,

$$d_p(x) = \lim_{\epsilon\to 0} \frac{\log \rho\left(B(x,\epsilon)\right)}{\log \epsilon}\ , \tag{1}$$

where $\rho\left(B(x,\epsilon)\right)$ denotes the ρ-measure (one can imagine the measure as specifying a shade of gray between black and white) of a ball of radius ϵ centered at x, and $d_p(x)$ is called the pointwise dimension of K relative to x. Now define the "sub-fractal",

$$K_\alpha = \left(x\epsilon K \middle| d_p(x) = \alpha\right)\ , \tag{2}$$

which is the set of points x from K relative to which the pointwise dimension value α is taken. Less restrictive possibilities than eq. (1) also exist, allowing for possibly greater generality in construction of K_α for some cases. In general, the collection of all these "sub-fractal" K_α's, for $\alpha \geq 0$, may be thought of as a "multi-fractal." This notion first appeared in Ref. (1) for the context of modeling fluid turbulence. The remarkable theoretical scenario that $f(\alpha)$, where

$$f(\alpha) = \dim K_\alpha, \tag{3}$$

is a smooth function of α, despite the decidedly non-smooth properties of the K_α and K, was laid out in Ref. 2. Moreover, $f(\alpha)$ was argued to have a variety of special properties: (1) it is everywhere concave downwards; (2) its peak value is dim K; (3) $f(\alpha)$ intersects the α-axis with infinite slope, at positive and finite values; and (4) the line $f(\alpha) = \alpha$ is tangent to $f(\alpha)$ where f and α are equal, and this value is the information dimension of K (or the dimension of the measure ρ). The $f(\alpha)$ formalism has been used with success to model data in several contexts; $f(\alpha)$ curves with one or more of the basic expected properties violated have been found; and the scheme has received widespread application as a means of organizing fractal data.

We have the first rigorous proofs of all the results described above for generalized Cantor sets (Moran fractals[3], with the product measure ρ defined below.) And, we have proofs of when the limit of eq. (1) exists, with answers to some open, hitherto unanalyzed issues. In particular, is the collection of all K_α equal to K? In other words, does the multi-fractal procedure get back the whole fractal? It assuredly does not; however, in the sense of ρ-measure, it does,

$$\rho\left(\bigcup_\alpha K_\alpha\right) = \rho(K)\ . \tag{4}$$

Finally, we have an example where the "fractal" K is the unit interval, but the collection of the K_α is a Baire first category set (i.e. is topologically meager).

POSSIBLE MULTIFRACTAL GENERALIZATION

The way the $f(\alpha)$ curve is constructed for Moran fractals in $\mathbf{R}^m$ can be stated a little more precisely than has been possible in the early literature[4]. For a set $(t_1,\ldots,t_n)$ of contracting similarity ratios, let K be a Moran fractal constructed with these ratios from seed set J. (In the middle-thirds Cantor set, $n = 2$, $t_1 = t_2 = 1/3$ and $J = [0,1]$.) It is important to note for the standard middle-thirds prototype not only are the ratios fixed but the similarity maps implementing the construction are fixed. The latter need not be so for the general Moran case.

Now, fix a probability vector $(p_1,\ldots,p_n)$ and let ρ be the probability measure naturally defined on K via redistribution. In otherwords, $\rho(J_i) = p_i$, where J_i, $i=1$ to n, are the sets obtained from J by similarities with contraction ratios t_i. The J_i comprise the first generation of the construction of K. The sets obtained in successive generations of the construction are assigned probabilities which are products of the p_i's in the natural way (product measure). The starting point for the $f(\alpha)$ construction is the auxiliary measure μ_q, $q\in\mathbf{R}$, which is an infinite product measure but based on $(p_1^q t_1^{\beta(q)},\ldots,p_n^q t_n^{\beta(q)})$, where

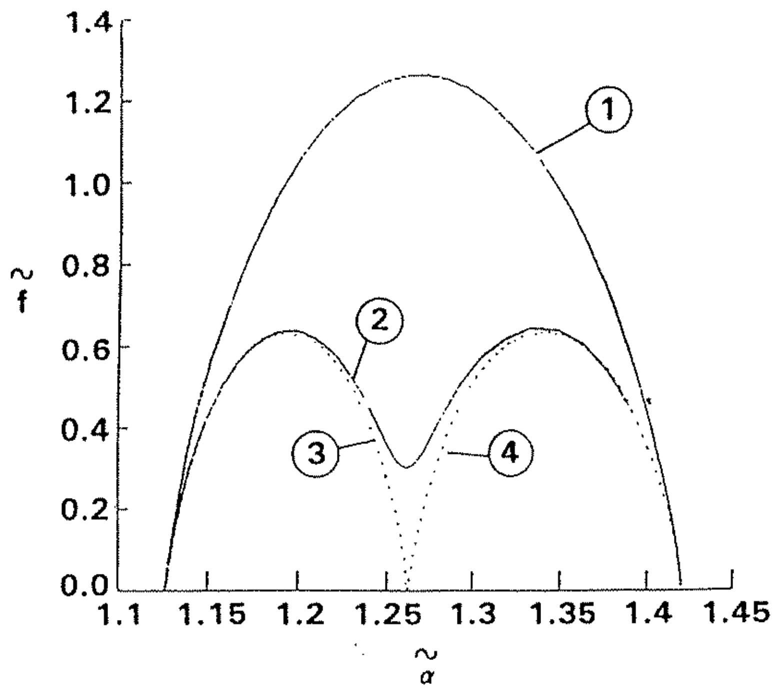

FIGURE 1. Generalized multifractal curves for a Moran construction.

$$\sum_{i=1}^{n} p_i^q t_i^{\beta(q)} = 1 . \qquad (5)$$

A slight generalization of this is usually referred to as the partition function owing to parallels to statistical mechanics. The formula for $f(\alpha)$ depends on ρ, on the fractal set, and on the function $\beta(q)$ uniquely solving eq. (5). We have proved the existence of a multi-fractal construction based on the generalized quantity $\tilde{\beta}(q,\underline{w})$ specified by the normalization of a slightly different probability vector

$$\sum_{i=1}^{n} w_i p_i^q t_i^{\tilde{\beta}(q,\underline{w})} , \; w_1 > 0, \; \ldots, \; w_n > 0 , \qquad (6)$$

and where $\underline{w}$ denotes the n-tuple $(w_1,\ldots,w_n)$. Note that $w_1=\ldots=w_n=1$ is the usual case: i.e. $\tilde{\beta}(q,\underline{1}) = \beta(q)$. The general properties of the $\tilde{f}(\tilde{\alpha}, \underline{w})$ curve that results are no longer those laid out above for the $f(\alpha)$ curve. We have several results about the generalized scheme. One of these is that the $\tilde{f}(\tilde{\alpha}, \underline{w})$ curve is stationary under variation of $\underline{w}$ at $w_1=\ldots w_n=1$. The $f(\alpha)$ curve is probably an absolute maximum, a conjecture confirmed by initial numerical studies.

In Fig. 1, we show a numerical study giving results of varying the w_i. The measure and fractal parameter values chosen were $n = 4$, $t_1=\ldots=t_4= 1/3$, $p_1= 0.29$, $p_2= p_3= 0.25$, $p_4= 0.21$. The weights are, for the curve marked: (1) $w_1=\ldots= w_4= 1$; (2) $w_1= w_2= 1$, $w_3= w_4= 0.01$; (3) $w_1= w_2= 1$, $w_3= w_4= 0$; and $w_1= w_2= 0$, $w_3= w_4= 1$. The last two, extreme cases are "forbidden" by the theorems; and case (3) (resp. (4)) is a horizontal translate of the $f(\alpha)$ curve for a middle-thirds Cantor set having p_1+p_2 (resp., p_3+p_4) normalized to one. Note in particular that only the first of the two permissible cases has given a curve concave downwards everywhere. Studies of the generalized multifractal theory are in progress. For example, we don't know yet whether the analogue of eq. (4) holds for a nontrival weight system; and connections to statistical mechanics have to be investigated.

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