THE BIRKHOFF CENTER AND ANALYTIC SETS

By R. Daniel Mauldin[1]

It is shown here how G. D. Birkhoff's notion of the center of a homeomorphism or flow naturally gives rise to an analytic set in a product space. It is shown that for a wide class of spaces this set is not a Borel set.

Let X be a locally compact separable metric space with complete metric d and let $H(X)$ be the space of autohomeomorphisms of X. The space $H(X)$ has a topology under which it is a complete separable metric group [6, 9]. For a wide class of X's, it is known that this topology is unique [7]. This topology may be briefly described as follows. Let $X^* = X \cup \{\infty\}$ be the one point compactification of X and consider the space $M = M(X^*, X^*)$ of all continuous maps of X^* into X^* provided with the compact open topology [9]. In this topology, M is a Polish space: M is separable and possesses a complete metric compatible with this topology. Identify $H(X)$ with $F = \{(f, g) \in M \times M : fg = gf = \mathrm{id}_{X^*}$ and $f(\infty) = \infty\}$. Since F is closed in $M \times M$, F is also a Polish space. We consider $H(X)$ to have this topology.

If $h \in H(X)$ and Y is an h-invariant subset of X, then a point $y \in Y$ is said to be nonwandering with respect to Y provided there is an increasing sequence of positive integers $n_1, n_2, n_3, \ldots$ and points $y_p \in Y$, $p = 1, 2, 3, \ldots$ such that the sequence $h^{n_p}(y_p)$ converges to y. Let $R_h(Y) = \{y \in Y : y$ is nonwandering with respect to $Y\}$. If Y is a closed h-invariant set, then $R_h(Y)$ is also closed and h-invariant. Set $R_h^0(X) = X$ and by recursion, for each ordinal α, $R_h^{\alpha+1}(X) = R_h(R_h^\alpha(X))$ and, if λ is a limit ordinal, $R_h^\lambda(X) = \bigcap_{\alpha<\lambda} R_h^\alpha(X)$. Since this "central" sequence $\{R_h^\alpha(X)\}$ forms a decreasing transfinite sequence of closed sets in X, there is a least countable ordinal $\delta = \delta(h)$ such that $R_h^{\delta+1}(X) = R_h^\delta(X)$. This ordinal is called the depth of h and $R_h^\delta(X) = R_h(X)$ is called the center of h. Of course, $R_h(X)$ is the closure of the set of all h-recurrent points.

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(A point x is h-recurrent means there is an increasing sequence of positive integers $n_1, n_2, n_3, \ldots$ such that the sequence $h^{n_p}(x)$ converges to x.)

The universal center of $H(X)$ as an analytic set. Let

$$(1.1) \qquad R = R(X) = \{(h, x) \in H(X) \times X : x \in R_h(X)\}.$$

Thus, $R = R(X)$ is the "universal" center of $H(X)$. For each ordinal α, let

$$(1.2) \qquad R^\alpha = \cup_{h \in H(X)} \{h\} \times R_h^\alpha(X).$$

Of course, $\{R^\alpha\}_{\alpha<\omega_1}$ is a decreasing transfinite sequence and $\cap_{\alpha<\omega_1} R^\alpha = R$.

THEOREM 1. *For each countable ordinal* α, R^α *is a Borel subset of* $H(X) \times X$.

Proof. It suffices to show that if R^α is a Borel set, then $R^{\alpha+1}$ is a Borel set. Let $\{V_n\}_{n=1}^\infty$ be a base for the topology of X. For positive integers m and n, set

$$(1.3) \qquad P(m, n) = \{h : h^{-m}(V_n \cap R_h^\alpha) \cap R_h^\alpha = \phi\}.$$

Then

$$(1.4) \qquad H(X) \backslash P(m, n) = \text{proj}_{H(X)} W(m, n),$$

where

$$(1.5) \quad W(m, n) = \{(h, x, y) : x \in R_h^\alpha, y \in R_h^\alpha \cap V_n \text{ and } y = h^m(x)\}.$$

Since the map $(h, x) \to h^m(x)$ is continuous, V_n is σ-compact, and R_h^α is closed in X, it follows that for each h, $W(m, n)_h$ is σ-compact. Thus, $\text{proj}_{H(X)} W(m, n)$ is a Borel set [14]. Let

$$(1.6) \qquad E(n) = \left[\bigcap_{m=1}^{\infty} P(m, n) \times V_n \right] \cap R^\alpha.$$

Since $R^{\alpha+1} = R^\alpha \backslash \cup_{n=1}^\infty E(n)$, $R^{\alpha+1}$ is a Borel set.

A different tact is taken to show R is an analytic set.

THEOREM 2. *The set R is an analytic subset of $H(X) \times X$.*

Proof. Let $B = \{(h, x): x \text{ is } h\text{-recurrent}\}$. Then B is a Borel subset of $H(X) \times X$. This may be seen by setting, for positive integers m and n,

$$(1.7) \qquad B(m, n) = \{(h, x): d(h^n(x), x) \leq 1/m\}.$$

If the sequence $\{(h_p, x_p)\}_{p=1}^{\infty}$ converges to the pair (h, x), then $\{h_p^n\}_{p=1}^{\infty}$ converges to the homeomorphism h. But, convergence in this topology implies continuous convergence [9]. This means $\{h_p^n(x_p)\}_{p=1}^{\infty}$ converges to $h(x)$. Thus, each set $B(m, n)$ is closed in $H(X) \times X$ and B is an $F_{\sigma\delta}$ set, since

$$(1.8) \qquad B = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B(m, k).$$

Now, R is the sectionwise closure of B:

$$(1.9) \qquad R = \bigcup_{h \in H(X)} \{h\} \times \mathrm{cl}_X(B_x).$$

But, the sectionwise closure E of an analytic set A in a product space $X \times Y$ is an analytic set:

$$(1.10) \quad E = \mathrm{proj}_{1,2}\{(x, y, y_1, y_2, y_3, \ldots) \in X \times Y \times Y^{\omega}: \forall n (x, y_n) \in A \text{ and } y_n \to y\}.$$

Coanalytic operators and the boundedness principle. In order to more carefully analyze the universal center set, the complement of R will be expressed as the set constructed from the empty set by a monotone, inductive, coanalytic ($=\Pi_1^1$) operator Γ [2].

Define $\Gamma: \mathcal{P}(H(X) \times X) \to \mathcal{P}(H(X) \times X)$ by $\Gamma(K) = K \cup \Psi(K)$, where

$$(h, x) \in \Psi(K) \leftrightarrow \forall(\langle n_p \rangle, \langle y_p \rangle) \in N^N \times X^N[\exists i (h, y_i) \in K \text{ or } \exists m \forall p\, d(x, h^{n_p}(y_p)) \geq 1/m].$$

If $A \subset B \subset X$, then $A \subset \Gamma(A)$ and $\Gamma(A) \subset \Gamma(B)$. Thus, Γ is monotone and inductive. Note that if $K \subset H(X) \times X$ and for each h, K_h is fully h-invariant, then $\Gamma(K)_h$ simply adds to K_h all the h wandering points of $X \backslash K$. The operator Γ constructs from $A \subset H(X) \times X$ a transfinite sequence $\{\Gamma^\alpha(A) : \alpha \in \mathrm{ORD}\}$ as follows:

$$\Gamma(A) = A, \tag{2.1}$$

$$\Gamma^{\alpha+1}(A) = \Gamma(\Gamma^\alpha(A)), \qquad \text{for all ordinals } \alpha, \tag{2.2}$$

$$\Gamma^\lambda(A) = \bigcup_{\alpha<\lambda} \Gamma^\alpha(A) \qquad \text{for limit ordinals } \lambda > 0. \tag{2.3}$$

Note that for each ordinal α, $\Gamma^\alpha(\phi) = (H(X) \times X) \backslash R^\alpha$ and $\Gamma^{\omega_1}(\phi) = (H(X) \times X) \backslash R$.

THEOREM 3. *The operator Γ is monotone, inductive and coanalytic.*

Proof. Since the union of two coanalytic operators is coanalytic, it suffices to show the operator Ψ is coanalytic [2]; i.e., there is a Polish space Y and a Borel operator Δ on $H(X) \times X \times Y$ such that for all (h, x) and K:

$$(h, x) \in \Psi(K) \leftrightarrow \forall y (h, x, y) \in \Delta(K \times Y). \tag{2.4}$$

Set $Y = N^N \times [H(X) \times X \times X]^N$. For each i, let $f_i(h, x, \langle n_p \rangle, \langle h_p, x_p, y_p \rangle) = (h, x_i, \langle n_p \rangle, \langle h_p, x_p, y_p \rangle)$. Let

$$G = \{(h, x, \langle n_p \rangle, \langle h_p, x_p, y_p \rangle) : \forall p h^{n_p} = h_p\} \tag{2.5}$$

$$M = \{(h, x, \langle n_p \rangle, \langle h_p, x_p, y_p \rangle) : \forall p h_p(x_p) = y_p\}, \tag{2.6}$$

and

$$D = \{(h, x, \langle n_p \rangle, \langle h_p, x_p, y_p \rangle) : \exists m \forall p d(x, y_p) \geq 1/m\}. \tag{2.7}$$

The set D is an F_σ set. Since the map $(h, x) \to h(x)$ is a continuous map of $H(X) \times X$ onto X, the set M is closed. Also, since composition map of $H(X) \times H(X)$ onto $H(X)$ is continuous the set G is closed. Define the operator Δ on $\mathcal{P}(H(X) \times X \times Y)$ by setting

$$(2.8) \quad \Delta(E) = \bigcup_{i=1}^{\infty} f_i^{-1}(E) \cup D \cup (H(X) \times X \times Y)\backslash(G \cup M).$$

Clearly, Δ is a Borel operator over $H(X) \times X \times Y$ and (2.4) holds.

THEOREM 4. *The set R is a Borel set if and only if there is some ordinal $\alpha < \omega_1$ such that the depth of each homeomorphism of X is $\leq\alpha$.*

Proof. If R is a Borel set, then $(H(X) \times X)\backslash R = \Gamma^{\omega_1}(\phi)$ is a Borel set. By the boundedness principle for such operators [2], there is a countable ordinal α such that $\Gamma^{\alpha}(\phi) = \Gamma^{\omega_1}(\phi)$. This means $R_h^{\alpha}(X) = R_h(X)$, for each $h \in H(X)$ and the depth of each homeomorphism is $\leq\alpha$. Conversely, if $\delta(h) \leq \alpha$, if each h, then $R^{\alpha} = R$, and R is a Borel set.

Remarks. One could have proven this last theorem by dualizing the operator Γ to obtain a derivation and using the boundedness principle for derivations [3]. Or, one could use a rank argument by considering the function $\varphi: R \to \omega_1$ given by $\varphi(h, x) = \min\{\alpha : (h, x) \in \Gamma^{\alpha}\phi\}$. The function φ is a coanalytic norm and one could use the boundedness principle for such norms [11].

Spaces with nonBorel centers. Let K be the Cantor space. Thus, K is a compact metrizable dense-in-itself, 0-dimensional space. It is known that an autohomeomorphism of a closed nowhere dense subset of K can be extended to a autohomeomorphism of K [5] [8]. We show here that there is an extension which has the same center. The proof is essentially a modification of van Engelen's argument for an extension [5]. Consequently, the proof is only outlined. This theorem is stated, but not proven, in van Douwen's manuscript [4].

THEOREM 5. Let A be a closed nowhere dense subset of the Cantor space, K and let h be an autohomeomorphism of A. There is an extension $\hat{h}$ of h to some autohomeomorphism of K such that $R_{\hat{h}}^1(K) \subset A$.

Proof. Fix a partition $\{V_n\}_{n=0}^{\infty}$ of $K\backslash A$ into nonempty, pairwise disjoint clopen sets such that

$$(3.1) \qquad \forall i \; \operatorname{diam}(V_i) < d(V_i, A),$$

and

$$(3.2) \qquad \lim_{i\to\infty} d(V_i, A) = 0.$$

The proof of the theorem is based upon the following lemma. We note that condition (3.7) insures the preservation of the center.

LEMMA 6. *There are bijections* ρ, $\sigma:\omega \to \omega$ *and a sequence* $\{a_n\}_{n=0}^{\infty}$ *of points of* A *with the properties*:

If n *is even*,

$$d(V_{\rho(n)}, a_n) < 2d(V_n, A), \tag{3.3}$$

and

$$V_{\sigma(n)} \subset B(h(a_n), d(V_{\rho(n)}, A)). \tag{3.4}$$

If n *is odd*:

$$d(V_{\sigma(n)}, h(a_n)) < 2d(V_{\sigma(n)}, A), \tag{3.5}$$

and

$$V_{\rho(n)} \subset B(a_n, d(V_{\sigma(n)}, A)). \tag{3.6}$$

Finally,

$$\rho^{-1}\sigma \text{ has no periodic points.} \tag{3.7}$$

Proof. Let $\rho(0) = 0$ and choose a_0 such that (3.3) holds. Let $S_0 = \{i: V_i \subset B(h(a_0), d(V_{\rho(0)}, A)\}$. Since S_0 is infinite, choose $\sigma(0) \in S_0$ with $\sigma(0) \neq \rho(0)$.

Suppose n is a positive integer and a_i, $\sigma(i)$, and $\rho(i)$ have been defined for $i < n$ such that (3.3)–(3.6) hold if $i < n$ and there do not exist distinct integers $i_1, \ldots, i_k$ all less than n such that

$$\begin{aligned} \sigma(i_1) &= \rho(i_2) \\ &\vdots \\ \sigma(i_{k-1}) &= \rho(i_k) \\ \sigma(i_k) &= \rho(i_1). \end{aligned} \tag{3.8}$$

If n is odd, let $\sigma(n) = \min \omega\backslash\{\sigma(i): i < n\}$ and choose a_n such that (3.5) holds. Let $S_n = \{i \in \omega: V_i \subset B(a_n, d(V_{\sigma(n)}, A))$. S_n is infinite. Choose $\rho(n) \in S_n\backslash\{\sigma(i): i < n\}$. Then (3.6) holds for n. Clearly, there do not exist distinct integers $i_1, \ldots, i_k$ all less than $n + 1$ such that (3.8) holds. The argument is similar if n is even. Thus, σ and ρ are bijections of ω and since (3.8) never holds, $\rho^{-1}\sigma$ has no periodic points.

Proof of Theorem 5. For each $n \in \omega$, let h_n be a homeomorphism of $V_{\rho(n)}$ onto $V_{\sigma(n)}$. Let $\hat{h} = h \cup \bigcup_{n\in\omega} h_n$. Clearly, $\hat{h}$ is a bijection of K which extends h and $\hat{h}$ and its inverse are continuous at each point of $K\backslash A$. It is well known that $\hat{h}$ is continuous.

Finally, if $x \in K\backslash A$, there is some n such that $x \in V_{\rho(n)}$. Then $h(x) \in V_{\sigma(n)} = V_{\rho(\rho^{-1}(\sigma(n)))}$. So, $h(h(x)) \in V_{\sigma(\rho^{-1}(\sigma(n)))}$. Consider the bijection of ω, $s = \rho^{-1}\sigma$. By induction, for each $k \geq 1$, $h^k(x) \in V_{\sigma(s^{k-1}(n))}$. Since $\rho^{-1}\sigma$ has no periodic points, $\lim_{k\to\infty} s^k(n) = \infty$. This implies x is a wandering point of h.

THEOREM 7. *For each ordinal* $\alpha < \omega_1$, *there is a homeomorphism* h *of* K *with depth* α.

Proof. In an, as yet, unpublished manuscript, Eric van Douwen showed that for each countable ordinal α, there is a countable closed subset A of K and an autohomeomorphism h of A with depth α. The extension of h given by Theorem 5 has the same property.

THEOREM 8. *The universal center*, $R(K)$, *of the Cantor set* K *is analytic but is not a Borel set.*

THEOREM 9. *The universal center*, $R(X)$, *of a* C^∞ *n-manifold with* $n \geq 3$ *is analytic but not a Borel set.*

Proof. This follows from a theorem of D. A. Neumann [13]. He showed that there are even flows on X of arbitrarily high order.

Remark. The exact relationship between the iterative stages in the construction of the center of a flow and the stages in the construction of the center of its time $t \neq 0$ homeomorphism seems to be unresolved. Of course, the final objects, the center of the flow and the center of the homeomorphism are the same [6].

Question. Must the universal center of a two dimensional manifold be a Borel set? It follows from known results, mentioned later, that the universal center of one dimensional manifolds and of some two dimensional manifolds is a Borel set.

The universal center for flows. There is a natural generalization of the preceding theorems concerning homeomorphisms of flows. Let $F(X)$ be the space of flows on X. Thus, $F(X)$ consists of all continuous maps $\varphi: R \times X \to X$ such that for each $t \in R$, $\varphi(t, \cdot)$ is an autohomeomorphism of X and $\varphi(s + t, x) = \varphi(s, \varphi(t, x))$. Again, $F(X)$ has a natural Polish topology. Regard $F(X)$ as a subset of $H(X)^R$, where $H(X)^R$ has its compact open topology [9]. In this topology, $H(X)^R$ is a Polish space and $F(X)$ is a G_δ subset of it.

If $\varphi \in F(X)$ and Y is an φ-invariant subset of X, then a point $y \in Y$ is said to be nonwandering with respect to Y provided there is a sequence of numbers $t_1, t_2, t_3, \ldots$ converging to ∞ and points $y_p \in Y$, $p = 1, 2, 3, \ldots$ such that the sequence $\varphi(t_p, y_p)$ converges to y. Let $R_\varphi(Y) = \{y \in Y: y$ is nonwandering with respect to $Y\}$. If Y is a closed h-invariant set, then $R_\varphi(Y)$ is also closed and φ-invariant. Set $R_\varphi^0(X) = X$ and by recursion, for each ordinal α, $R_\varphi^{\alpha+1}(X) = R_\varphi(R_\varphi^\alpha(X))$ and if λ is a limit ordinal, $R_\varphi^\lambda(X) = \bigcap_{\alpha<\lambda} R_\varphi^\alpha(X)$. Since this "central" sequence $\{R_\varphi^\alpha(X)\}$ forms a decreasing transfinite sequence of closed sets in X, there is a least countable ordinal $\delta = \delta(\varphi)$ such that $R_\varphi^{\delta+1}(X) = R_\varphi^\delta(X)$. This ordinal is called the depth of φ and $R_\varphi^\delta(X) = R_\varphi(X)$ is called the center of φ. Of course, $R_\varphi(X)$ is the closure of the set of all φ-recurrent points (or even the φ-Poisson stable points). (A point x is h-recurrent means there is a sequence of numbers $t_1, t_2, t_3, \ldots$ converging to ∞ such that the sequence $\varphi(t_p, x)$ converges to x.)

Let

$$R_F(X) = R(X) = \{(\varphi, x) \in F(X) \times X: x \in R_\varphi(X)\}.$$

Thus, $R_F(X)$ is the "universal" center of $F(X)$. In view of Neumann's theorem, we have

THEOREM 10. *The universal center $R_F(X)$ of the space of flows of a C^∞ n-manifold with $n \geq 3$ is analytic, but, it is not a Borel set.*

Question. Must the universal center for flows on a two dimensional manifold be a Borel set? A. J. Schwartz and E. S. Thomas showed that the depth of a flow on an orientable 2-manifold of finite genus has depth ≤ 2 [15]. D. A. Neumann showed that in the nonorientable case the depth is ≤ 3 [12]. Thus, for these manifolds the universal center is a Borel set. For one dimensional manifolds, even the depth of a map is ≤ 2 [16, 17].

Question. For each $\alpha < \omega_1$, is there a locally compact metric space (or even a manifold) such that the depth of the universal center for homeomorphisms (or flows) is exactly α?

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UNIVERSITY OF NORTH TEXAS

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