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THE ORDER TYPE OF THE SET OF PISOT NUMBERS

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Abstract. Set $a_1 = \omega + 1 + \omega^*$ and for each positive integer n, set $a_{n+1} = a_n\omega + 1 + (a_n\omega)^*$. We show the order type of S, the set of Pisot-Vijayaraghavan numbers, is the ordered sum, $\sum_{n=1}^{\infty} a_n$.

Let S be the set of Pisot (or Pisot-Vijayaraghavan) numbers. Thus, S is the set of all algebraic integers $\theta > 1$ all of whose other conjugates lie inside the unit circle. This remarkable closed countable set has many interesting topological and analytic features. For example, the Cantor-Bendixson derived set order of S has been known for some time. To see this, we recall that Dufresnoy and Pisot showed that the minimal element of the nth derived set, $S^{(n)}$, is greater than $n^{1/4}$. On the other hand, the best result concerning upper bounds of $\min S^{(n)}$ seems to be one of Bertin [B]. She showed that $k \in S^{(2k-2)}$, for $k > 1$. It follows from these facts that the Cantor-Bendixson derived set order of S is ω. In this note, we make some observations which yield a characterization of one more facet of the topological distribution of S, the order type of S. This question was raised by Mauldin [MR,Problem 1071]. We make some notation: set $a_1 = \omega + 1 + \omega^*$ and for each positive integer n, set $a_{n+1} = a_n\omega + 1 + (a_n\omega)^*$. The order type of S is given in the last theorem of this note:

Theorem 6. *The order type of S is the ordered sum, $\sum_{n=1}^{\infty} a_n$.*

In order to prove this theorem, we need the fact that each element of $S^{(n)}$ is a limit from both sides of elements of $S^{(n-1)}$. We first present a proof of this fact in some detail.

Given a Pisot number θ, let $P(z)$ be its minimal polynomial, so $P(z)$ is an irreducible monic polynomial with integer coefficients having $P(\theta) = 0$ and such that all other roots of $P(z)$ lie in $|z| < 1$. All roots of $P(z)$ are simple and θ is its unique root in the interval $(1, \infty)$ so $P(1) < 0$. We will write $Q(z)$ for the reciprocal of $P(z)$, i.e. $Q(z) = z^{deg(P)}P(1/z)$, and hence $Q(0) = 1$, $Q(1) < 0$ and $Q(z)$ has a unique root in $|z| < 1$, namely $1/\theta$, with all other roots being in $|z| > 1$.

Let $\mathcal{C}$ denote the set of rational functions $f(z) = A(z)/Q(z)$, where A and Q are polynomials with integer coefficients, Q is the reciprocal of a minimal polynomial

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of a Pisot number θ, $A(0) \neq 0$, $A(1/\theta) \neq 0$, and $|A(z)| \leq |Q(z)|$ on $|z| = 1$. Thus $|f(z)| \leq 1$ on $|z| = 1$ and $f(z)$ has a unique pole in $|z| < 1$, this pole being a simple pole at $1/\theta$. Give $\mathcal{C}$ the topology of uniform convergence on compact subsets of the sphere. Then subsets of $\mathcal{C}$ corresponding to bounded sets of θ are compact. (See Theorem 2.2.1 of [BD]). Corresponding to each Pisot number there are (usually many) f in $\mathcal{C}$. The mapping of $\mathcal{C}$ to S defined by $f \to \theta$ is continuous.

If $\theta \in S^{(n)}$ then there is an $f \in \mathcal{C}^{(n)}$ with pole $1/\theta$. The set $\mathcal{C}'$ was characterized by Dufresnoy and Pisot [DP] as the set of $f \in \mathcal{C}$ for which $|f(z)| < 1$ for all but a finite subset of $|z| = 1$. Thus the isolated points of $\mathcal{C}$ consist of those f for which $|f(z)| = 1$ everywhere on $|z| = 1$. For these f, $A(z) = \pm P(z)$. For $n \geq 2$, the set $\mathcal{C}^{(n)}$ was characterized by Grandet-Hugot [GH,p.20]. The following notation is used: Given $n \geq 1$, let $N = \{1, 2, \ldots, n-1\}$, and if $(m_1, \ldots, m_{n-1})$ is a vector of integers, let $M(I) = \sum_{i \in I} m_i$, for any subset $I \subset N$.

Theorem 1 (Grandet-Hugot). *In order for $A/Q \in \mathcal{C}^{(n)}$, it is necessary and sufficient that there exist polynomials $B_I(z)$, $C_I(z)$ with integer coefficients, indexed by the subsets of N with $B_\emptyset = A$, $C_\emptyset = Q$, having the following properties:*

(1) For each $j \in N$, there is a subset $J \subset N$ with $j = \max J$ such that at least one of B_J or C_J is not identically zero.

(2) For all $|z| = 1$, the inequalities $|B_I(z)| \leq |Q(z)|$ and $|C_I(z)| \leq |Q(z)|$ hold, with equality for at most a finite set of z, (except for $C_\emptyset = Q$).

(3) For each vector of positive integers $(m_1, \ldots, m_{n-1})$, define $B(z) = \sum_{I \subset N} z^{M(I)} B_I(z)$ and $C(z) = \sum_{I \subset N} z^{M(I)} C_I(z)$. Then the rational function $B/C \in \mathcal{C}'$.

The condition (3) of this theorem is stated somewhat differently in [GH] but can be deduced from the proof given there. Note that it is quite possible for B and C in (3) to have a common factor.

We begin with a short discussion of the equation $Q_m(z) = Q(z) + z^m A(z)$, where $A/Q \in \mathcal{C}'$, following [BP]. In addition to the Pisot numbers, this requires consideration of the Salem numbers which are those algebraic integers $\theta > 1$ all of whose other conjugates lie in the closed unit disk $|z| \leq 1$ with at least one conjugate on $|z| = 1$. Let $0 \leq t < 1$. Then, by Rouché's theorem, for all $m \geq 0$, $Q(z) + tz^m A(z)$ has a unique root in the open unit disk. This root, $z(t)$, is clearly real and non-zero. Since it is a continuous function of t and $z(0) = 1/\theta > 0$ it follows that $0 < z(t) < 1$. As $t \to 1$, $z(t)$ tends to a root $0 < z(1) \leq 1$ of $Q_m(z)$ which we denote $1/\theta_m$. If $\theta_m > 1$ then $1/\theta_m$ is the unique root of $Q_m(z)$ in $|z| < 1$. Otherwise $Q_m(1) = 0$ and $Q_m(z)$ has no roots in $|z| < 1$. The polynomial $Q_m(z)$ may also have other roots on $|z| = 1$ at points where $|Q(z)| = |A(z)|$. These will be roots of the polynomial $\Omega(z) = z^r(Q(z)Q(1/z) - A(z)A(1/z))$, where $r > 0$ is chosen so that Ω is a polynomial with $\Omega(0) \neq 0$. The roots on $|z| = 1$ are necessarily simple except that if $\theta_m = 1$ then $z = 1$ may be a triple (but never a double) root.

The root inside the unit disk, if it occurs, is thus of the form $1/\theta_m$, where θ_m is either a Pisot or a Salem number. This follows from the fact that all of the conjugates of $1/\theta_m$ lie in $|z| \geq 1$. The roots of $Q_m(z)$ on $|z| = 1$ are either roots of unity or possibly conjugates of $1/\theta_m$ if θ_m is a Salem number.

It is not hard to see that $\theta_m > 1$ for sufficiently large m. For $Q_m(0) = 1$ and since $|A(1)| \leq |Q(1)| = -Q(1)$ we have $Q_m(1) = A(1) + Q(1) \leq 0$. Thus, if $A(1) < -Q(1)$ then $Q_m(1) < 0$ and hence $Q_m(z)$ has a root in $0 < z < 1$ for each

$m \geq 0$, i.e. $\theta_m > 1$ for any m in this case. On the other hand, if $A(1) = -Q(1)$, so $Q_m(1) = 0$, then $Q_m(z)$ will have a root in $0 < z < 1$ if the derivative $Q'_m(1) > 0$, and this holds as soon as $m > (-Q'(1) - A'(1))/A(1)$.

It is easy to see that if $1/\theta < 1$ is the root in $|z| < 1$ of $Q(z)$ then $\theta_m \to \theta$ as $m \to \infty$. Also, the numbers θ_m are eventually distinct since a common root of $Q_m(z)$ and $Q_n(z)$ would be a root of $(z^m - z^n)A(z)$, and $A(z)$ is non-zero in a neighbourhood of $1/\theta$ since $A(1/\theta) \neq 0$. Furthermore, θ_m must eventually be a Pisot number and not a Salem number. For, if θ_m is a Salem number then its conjugates on $|z| = 1$ are roots of the fixed polynomial Ω and hence θ_m is also a root of Ω. This can only occur for a finite set of m. In the following proof, we will need the following more precise result from [BP].

Lemma 2. *Suppose that $A/Q \in \mathcal{C}'$, $m > 1$, $m \neq deg(Q) - deg(A)$ and that $\theta_m > 1$. Then θ_m is a Pisot number.*

As a consequence of Theorem 1 and Lemma 2, we have the following result, stated on p.24 of [GH], with the condition "for all sufficiently large m" omitted, and with the remark that "it follows from the preceding proof". We give more details of the proof here.

Theorem 3. *If $A/Q \in \mathcal{C}^{(n)}$, for $n \geq 1$, and if $Q_m(z) = Q(z) + z^m A(z)$, for each positive integer m, then, for all sufficiently large m, $Q_m(z)$ has a root $1/\theta_m < 1$ for which $\theta_m \in S^{(n-1)}$.*

Proof. By the above discussion, there is an M_0 such that $m \geq M_0$ implies that $\theta_m \in S$. We must show that there is an $M''_0 \geq M_0$ for which $\theta_m \in S^{(n-1)}$ if $m \geq M''_0$. Let m be fixed with $m \geq M_0$.

Given a vector of positive integers $(m_1, \ldots, m_{n-1})$, let $B(z)$ and $C(z)$ be as in (3) so that $B/C \in \mathcal{C}'$. As in the discussion preceeding Lemma 2, $C(z) + z^m B(z)$ has at most one root in $|z| < 1$ and if this root exists, then it is real and positive. If this root exists, we denote its reciprocal by $\theta(m_1, \ldots, m_{n-1})$, otherwise we write $\theta(m_1, \ldots, m_{n-1}) = 1$. If $\theta(m_1, \ldots, m_{n-1}) > 1$ then it is a Pisot or a Salem number. We will denote $C(z) + z^m B(z) = R_n(m_1, \ldots, m_{n-1})$ whenever it is necessary to indicate the dependence on n and $m_1, \ldots, m_{n-1}$.

We are going to let $m_{n-1}, \ldots, m_1$ tend to ∞ in the order just listed. We must insure that we are dealing at each stage with a sequence of eventually distinct elements of S.

In order to insure that the $\theta(m_1, \ldots, m_{n-1})$ are Pisot numbers and not Salem numbers, it suffices by Lemma 2 to have $m > 1$ and $m + \deg(B) \neq \deg(C)$. This latter condition will require restrictions on m_k of the form $m_k \geq M'_k(m, m_1, \ldots, m_{k-1})$. For uniformity, define $m_0 = m$ and let K denote the set $\{0, 1, \ldots, n-1\}$. Also, if $I \subset K$ write $D_I = C_I$ if $0 \notin I$ and $D_I = B_J$ if $0 \in I = \{0\} \cup J$. Then $C + z^m B = \sum_{I \subset K} z^{M(I)} D_I$. It will be enough to show that we can restrict $(m_0, m_1, \ldots, m_{n-1})$ so that all of the non-zero terms of this sum have distinct degrees, that is, $M(I) + \deg(D_I)$ with $D_I \neq 0$ should be distinct. For, in this case

$$\deg(C) = \max_{I \subset N}(M(I) + \deg(C_I)) \neq m + \deg(B) = \max_{I \subset N}(m_0 + M(I) + \deg(B_I)).$$

We now show how to insure that the $M(I) + \deg(D_I)$ are distinct. Given $I \neq J \subset K$ with D_I and D_J non-zero, let k be the largest element in the symmetric difference $(I \backslash J) \cup (J \backslash I)$. Assume that $k \in I$ without loss of generality. Then $M(I) - M(J) = m_k + L(m_0, \ldots, m_{k-1})$, where L is a linear combination of $m_0, \ldots, m_{k-1}$ with coefficients in $\{-1, 0, 1\}$. Thus we will insist that $m_k > deg(D_J) - deg(D_I) - L(m_0, \ldots, m_{k-1})$ for each such I and J, giving a restriction $m_k \geq M'_k$, say, for $k \geq 0$. For $m = m_0$, this amounts to the restriction that $m \neq \deg(C_I) - deg(B_I)$ for any $I \subset N$ with both C_I and B_I non-zero. We also insist that $m > 1$.

In order to insure that the dependence of $C(z) + z^m B(z)$ on each of $m_1, \ldots, m_{n-1}$ is non-trivial, we must make some further restrictions on m. Notice that if $I \subset N$ and if both B_I and C_I are non-zero, there is at most one value of m for which $C_I + z^m B_I$ is identically zero. We omit this finite set of m from consideration, by taking $m \geq M''_0 \geq M'_0$, say. With this restriction on m and by (1) of Theorem 1, for each $j \in N$, there is a $J = J(j) \subset N$ with $j = \max J$ so that $C_J(z) + z^m B_J(z)$ is not identically zero. This insures the nontrivial dependence of $C(z) + z^m B(z)$ on m_j.

Now we are ready to consider the convergence of $\theta(m_1, \ldots, m_{n-1})$ to θ_m. We begin with $n = 1$, so that $R_1(m_1) = Q + z^{m_1} C_{\{1\}} + z^m(A + z^{m_1} B_{\{1\}})$ has the root $1/\theta(m_1)$. We observe that $\lim_{m_1 \to \infty} \theta(m_1) = \theta_m$. Since $\theta_m > 1$, we have $\theta(m_1) > 1$ for $m_1 \geq M_1$, say, and then $\theta(m_1) \in S$ for $m_1 \geq M'_1 \geq M_1$. As discussed above, the existence of $J(1)$ and the assumption $m \geq M''_0$ insures that the $\theta(m_1)$ for $m_1 \geq M''_1$, say, are distinct.

Similarly, for each $m_1 \geq M''_1$, $R_2(m_1, m_2)$ has a root $1/\theta(m_1, m_2)$ for which

$$\lim_{m_2 \to \infty} \theta(m_1, m_2) = \theta(m_1),$$

and then

$$\lim_{m_1 \to \infty} \lim_{m_2 \to \infty} \theta(m_1, m_2) := \lim_{m_1 \to \infty} (\lim_{m_2 \to \infty} \theta(m_1, m_2)) = \theta_m.$$

Again, the terms of the sequence are distinct elements of S for $m_2 \geq M''_2$, say.

By induction, we have the iterated limit

$$\lim_{m_1 \to \infty} \cdots \lim_{m_{n-1} \to \infty} \theta(m_1, \ldots, m_{n-1}) = \theta_m,$$

where at each stage we are dealing with a sequence of eventually distinct elements of S. This shows that $\theta_m \in S^{(n-1)}$, for all $m \geq M''_0$. $\square$

Remark. The sequence $\theta(m_1, \ldots, m_{n-1})$ is as considered in [GH], where it is asserted that $\theta(m_1, \ldots, m_{n-1}) \in S$ and that $\theta_m \in S^{(n-1)}$ without the requirement that m be sufficiently large. As our proof shows, there are three possible complications. The first is that $\theta(m_1, \ldots, m_{n-1}) = 1$ is possible. For example, this occurs for $1/(1 - 2z)$ with $m = 1$. This is easily avoided by the requirement $m \geq M_0$.

A more serious complication is that $\theta(m_1, \ldots, m_{n-1})$ may depend trivially on some of the parameters and this means that θ_m may be in S but fail to be in $S^{(n-1)}$. For example, in [B], it is shown that $1/(1 - 2z - z^2) \in \mathcal{C}^{(3)}$, but $1 - z - z^2 = 1 - 2z - z^2 + z^1$ defines only an element of $S^{(1)}$ not $S^{(2)}$. Our proof shows that this occurs only for a finite set of m.

The other main complication is caused by the fact that $\theta(m_1, \ldots, m_{n-1}) > 1$ may be a Salem number rather than a Pisot number. The possibility that $\theta_m > 1$ may be a Salem number was first pointed out by Walter Parry. The example $A(z) = 1 - z^2$, $Q(z) = 1 - 2z - z^2 + z^4$, $m = 1$ given in [BP] is due to him. Theorem 1 of that paper shows that in fact *every* Salem number satisfies such an equation. Theorem 2 of that paper states that this is only possible for $m = 1$ but only the case $m + \deg(A) \neq \deg(Q)$ is proved there. Since the proof of the remaining case $m + \deg(A) = \deg(Q)$ has not yet appeared, we do not rely on it in the proof of Theorem 3, even though that would simplify the proof considerably: the conditions $m_k \geq M'_k$ required to insure $m + \deg(B) \neq \deg(C)$ could be replaced by the simple condition $m > 1$.

Corollary 4. *If $\theta \in S^{(n)}$ for some $n \geq 1$, then θ is a two-sided limit of elements of $S^{(n-1)}$.*

Proof. Let $A/Q \in \mathcal{C}^{(n)}$ with pole at $1/\theta$. Then also $-A/Q \in \mathcal{C}^{(n)}$. By Theorem 2, for all but a finite set of m, $Q_m^{\pm}(z) := Q(z) \pm z^m A(z)$ defines an element $\theta_m^{\pm} \in S^{(n-1)}$. Since $Q_m^{\pm}(1/\theta) = \pm\theta^{-m} A(1/\theta)$, the numbers θ_m^{+} and θ_m^{-} lie on opposite sides of θ, and hence θ is a limit from both sides of elements of $S^{(n-1)}$. □

Lemma 5. *Let z be an isolated point of $S^{(n)}$ and $a < z < b$ be such that $S^{(n)} \bigcap (a, b) = z$ and $a, b \notin S$. Then the order type of $S \bigcap (a, b)$ is a_n.*

Proof. Let $c_1, c_2, c_3, \ldots$ be an increasing sequence consisting of the elements of $S^{(n-1)}$ in (a, z) and let $d_1, d_2, d_3, \ldots$ be a decreasing sequence consisting of the elements of $S^{(n-1)}$ in (z, b). If $n = 1$, then clearly the order type of $S \bigcap (a, b)$ is a_1. Suppose the lemma holds for n. Let $a = u_0 < c_1 < u_1 < \ldots < u_{k-1} < c_k < u_k < \ldots$ with each u_k not in S. Then by the induction hypothesis, the order type of $S \bigcap (u_{k-1}, u_k)$ is a_n for each k. Therefore, the order type of $S \bigcap (a, z)$ is $a_n\omega$. Putting this together with a similar argument for $S \bigcap (z, c)$, we have the order type of $S \bigcap (a, b)$ is a_{n+1}. □

Let $x_n = \min S^{(n)}$. The sequence x_n is strictly increasing and it is known that x_0 is the real root of $x^3 - x - 1$, $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = 2$. In [Bo,p.7] there is an explicit conjecture as to the value of x_n for each $n > 2$.

Theorem 6. *The order type of S is the ordered sum, $\sum_{n=1}^{\infty} a_n$.*

Proof. For each n, choose y_n not in S with $x_n < y_n < x_{n+1}$. Set $D_1 = S \bigcap [x_0, y_1)$ and for each $n > 1$, $D_n = S \bigcap (y_{n-1}, y_n)$. Then the order type of S is the ordered sum of the order types of the sets D_n. By lemma 4, the order type of each D_n is a_n. □

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