Generic properties of open billiards

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Abstract - The purpose of this paper is to show that for a dense G_δ set of three smooth convex bodies with nowhere vanishing curvature (in the C^k topology, $2 \leq k \leq \infty$), the open billiard obtained from these convex bodies determines a potential (the one that defines the natural escape measure of this billiard) which is non-lattice. This result generalizes one of the results obtained in a previous work of A. Lopes and R. Markarian [1].

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1. The open billiard

The open billiard was previously analyzed in [1]. We refer the reader to [1] for most of the results we will use in the present paper. Most of the theorems of a dynamical nature mentioned in this paper [1] are stated for the open billiard defined by three circles with the same radius, but as was mentioned in [1] (see end of section 1), it can be easily extended to general convex bodies satisfying Morita's condition [2]. However, the proof of the result stated in section 8 [1] about the non-lattice property of the natural potential cannot be directly adapted from [1] to the general case. The purpose of the present paper is to eliminate this gap.

We refer the reader to [5] for a general reference for billiards.

We assume that the open billiard is defined by three convex scatterers or bounded convex domains O_1, O_2 and O_3 in $\mathbf{R^2}$ each with a class $C^k, 2 \leq k \leq \infty$, (see [4] for definitions) boundary, and each with nonvanishing curvature everywhere. Let $\mathcal{F}$ be the space of all such curves. The space $\mathcal{F}$ carries a natural topology which we call the C^k topology under which it is a complete separable metric space (see [4]). We will assume that the open billiard is defined by three curves, implicitly given respectively by three $C^k, 2 \leq k \leq \infty$, expressions

$$f(x, y) = 0,$$

$$g(x, y) = 0$$

and

$$h(x, y) = 0,$$

that is $f, g, h \in \mathcal{F}$, where $\mathcal{F}$ is the set of C^k functions of $\mathbf{R^2}$ in $\mathbf{R}$.

Let F be the subset of $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$ consisting such that for $(f, g, h) \in F$, the three curves $\gamma_1, \gamma_2, \gamma_3$ implicitly defined by the above equations are smooth Jordan curves and define convex bodies. We also assume all (O_1, O_2, O_3) satisfy Morita's condition [2]: the convex hull of any two of these bodies do not intersect the third one. The set F is a G_δ subset of $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$.

For each $(O_1, O_2, O_3) \in F$, we consider the associated map $T_{(O_1, O_2, O_3)} = T$ restricted to the boundary values, i.e., position q and angle ϕ, of the billiard as in [1].

The two-dimensional map T associated to the boundary points $x = (q, \phi)$ is hyperbolic when restricted to the Cantor set Π consisting of those x that do not escape to infinity [1]. The dynamical system T will be therefore defined from Π to itself. There is a natural measure or escape measure, μ, for this system. The escape measure μ has the following intuitive description. Consider in the plane a certain expanding transformation whose non-wandering set is a Cantor set with Lebesgue measure zero. A natural generalization of the Bowen-Ruelle-Sinai measure in this case might be obtained in the following way. Given a set B contained in the Cantor set C, we are going to define the value $\mu(B)$. Consider a grid of squares with side ϵ . Denote by b_ϵ the number of squares that intersect B and c_ϵ the number of squares that intersect the Cantor set C. Now, when ϵ goes to zero, if the limit

$$\lim_{\epsilon \to 0} \frac{b_\epsilon}{c_\epsilon} = \mu(B)$$

exists and if this limit is independent of the grid for any Borel set B, then we say that μ is a "natural" (or escape) measure. This procedure is quite natural from the point of view of an experimental observer. Given what is left after n observations (this will produce a slightly distorted grid with a value ϵ inversely proportional to n), then one should consider the proportion of what is left of the set that one wants to measure over the full set that still remains. The role of the grid is to give a computable approximation of the Lebesgue measure.

The measure μ, we consider here is obtained as a limit of the above procedure. An important fact is that μ is also the unique equilibrium state of the natural potential, ψ (see [1]). We will give the expression for ψ a little later. But, first let us briefly recall the meaning of equilibrium state. Given a measurable transformation $T : M \to M$ on a measurable space $(M, \mathcal{F})$ and a function $\psi(x)$ defined on the space M, define the Topological Pressure of ψ by

$$P(\psi) := \sup\{h(\nu) + \int \psi(x) d\nu(x)\},$$

where the sup is taken over all invariant probabilities ν. A measure θ is called an equilibrium measure for ψ if

$$P(\psi) = h(\theta) + \int \psi(x) d\theta(x).$$

Let us mention that the potential considered by Morita in [2] is not the natural potential but rather the ceiling function is considered as the potential in that paper. The equilibrium measure generated by the ceiling function is not the same as the escape measure. Therefore, the questions addressed in [1] and here are of a different nature than the those considered in [2].

A function Θ defined on Π is called non-lattice if there does not exist a function v, a constant α and a function G taking only integers values, such that for all $x \in \Pi$

$$v(T(x)) - v(x) + \alpha G(x) = \Theta(x).$$

Non-lattice functions defined on the non-wandering set of a hyperbolic dynamical system T determine nice statistical properties of the dynamical zeta function associated to the periodic orbits of T [3].

In [2], Morita shows that the ceiling potential is not lattice. We denote the ceiling function by $t(x)$ in this note.

One of the results obtained in [1] is that for a dense set of values $a > 2$, open billiards determined by three circles of radius one centered in the vertices of an equilateral triangle of side a, satisfy the following property: the associated natural potential ψ is non-lattice.

In this paper we prove

Theorem 1: For a dense G_δ set of parameters $(O_1, O_2, O_3) \in F$ in the C^k topology, $2 \leq k \leq \infty$, the open billiard defined by (O_1, O_2, O_3) is such that the natural potential ψ is non-lattice.

Note that the results of [1] do not follow from the above theorem, because the perturbations allowed here can leave the class of circular billiards.

2. Proof

Let us outline the fundamental ideas of the proof: (1) a lattice potential must satisfy the condition that its time averages over all periodic orbits are rationally related, (2) the natural potential on a given periodic orbit depends only on the dynamics near that orbit, and (3) the scatterers can be deformed so as to perturb one periodic orbit while leaving another periodic orbit (and nearby trajectories) unchanged. Now we will prove the main theorem.

Proof of Theorem 1 : Consider periodic orbits of period respectively 2 and 3 for T denoted by $a_1, a_2 \in \Pi$ and $b_1, b_2, b_3 \in \Pi$.

The proof proceeds by way of contradiction.

So, suppose that there exist a function v, a constant $\alpha \in \mathbf{R}$ and an integer valued function G such that $v \circ T - v + \alpha G = \psi$, where ψ is the natural potential for the escape measure. Then

$$\psi(a_1) + \psi(a_2) = \psi(a_1) + \psi(T(a_1)) =$$

$$(v(a_2) - v(a_1) + \alpha G(a_1)) + (v(a_1) - v(a_2) + \alpha G(a_2)) = m_1 \alpha$$

for some $m_1 \in \mathbf{Z}$. Similarly, we have

$$\psi(b_1) + \psi(b_2) + \psi(b_3) = n_1 \alpha$$

for some $n_1 \in \mathbf{Z}$. Therefore,

$$\frac{1}{m_1}(\psi(a_1) + \psi(a_2)) = \frac{1}{n_1}(\psi(b_1) + \psi(b_2) + \psi(b_3)). \tag{1}$$

Now we need to use the analytic expression of ψ. Recall from [1] that $\phi(x)$ denotes the angle with the normal of the trajectory beginning at $x = (q, \phi)$ and $K(x) = K(q)$ is the curvature at q of the curve γ (one of the components of the boundary of the billiard) such that $q \in \gamma$. From [5], ψ is given by

$$\psi(x) = \log|1 + t(x)k(x)|,$$

for $x \in \Pi$, where $t(x) = \|q - q'\|$ is the distance between the successive hits $x = (q, \phi)$ and $T(x) = (q', \phi')$, and $k(x)$ is given by the continued fraction, $k(x) = [c_1(x), c_2(x), c_3(x), \ldots]$ or

$$k(x) = c_1(x) + \cfrac{1}{c_2(x) + \cfrac{1}{c_3(x) + \cfrac{1}{c_4(x) + \cdots}}}$$

where

$$c_{2k+1}(x) = \frac{2K(x)}{\cos\phi(T^{-k}(x))}\ ,\ c_{2k}(x) = t(T^{-k}(x))\ , k \in \mathbf{N}.$$

Expression (1) can be rewritten as

$$((1+t(a_1)k(a_1))(1+t(a_2)k(a_2)))^{n_1} =$$

$$((1+t(b_1)k(b_1))(1+t(b_2)k(b_2))(1+t(b_3)k(b_3)))^{m_1}. \qquad (2)$$

We point out that the values a_1, a_2 defining the orbit of period 2 and the values b_1, b_2, b_3 defining the orbit of period 3 depend continuously on (O_1, O_2, O_3). Note that $t(a_i), i \in \{1,2\}, t(b_j), j \in \{1,2,3\}$ are continuous functions of (O_1, O_2, O_3). Finally, note also that $\phi(a_i), \phi(b_j)$ and $K(a_i), K(b_j)$ are continuous functions of (O_1, O_2, O_3). Therefore, all these values t, K, ϕ and also $c_i, i \in \mathbf{N}$ are continuous functions of (O_1, O_2, O_3).

We claim that $k(a_i), i \in \{1,2\}$ and $k(b_j), j \in \{1,2,3\}$ are also continuous functions of (O_1, O_2, O_3). In order to prove the claim, note that from the periodicity of a_1 and a_2

$$k(a_1) = c_1(a_1) + \cfrac{1}{c_2(a_1) + \cfrac{1}{c_3(a_1) + \cfrac{1}{c_4(a_1) + \cfrac{1}{k(a_1) + \cdots}}}},$$

or $k(a_1) = \overline{[c_1(a_1), c_2(a_1), c_3(a_1), c_4(a_1)]}$ is a periodic continued fraction. Therefore, $k(a_1)$ is a solution of a quadratic equation with coefficients in

$$c_1(a_1), c_2(a_1), c_3(a_1), c_4(a_1).$$

Similarly, the same property also holds for $k(a_2)$.

Finally, from the periodicity of b_1, b_2, and b_3, the value $k(b_1)$ is also a solution of a quadratic equation with coefficients in

$$c_1(b_1), c_2(b_1), c_3(b_1), c_4(b_1), c_5(b_1), c_6(b_1)$$

since

$$k(b_1) = \overline{[c_1(b_1), c_2(b_1), c_3(b_1), c_4(b_1), c_5(b_1), c_6(b_1)]}.$$

Therefore, the terms in (2) depend in a continuous fashion on (O_1, O_2, O_3). Thus, for a fixed m_1, n_1, the set B_{m_1,n_1} consisting of all $(O_1, O_2, O_3) \in F$ such that (2) holds is a closed set in F.

We now show that for fixed m_1, n_1 this set B_{m_1,n_1} is nowhere dense in F. In order to do that we will show that for $(O_1, O_2, O_3) \in B_{m_1,n_1}$ one can perturb the three curves in F changing the value

$$((1 + t(a_1)k(a_1))(1 + t(a_2)k(a_2)))^{n_1}$$

without changing the period three orbit and also without changing

$$((1 + t(b_1)k(b_1))(1 + t(b_2)k(b_2))(1 + t(b_3)k(b_3)))^{m_1}.$$

Geometrical arguments easily show that one can perturb just the period two orbit (without changing the period three orbit at all) by changing a little bit the value $t(a_1) = t(a_2)$ and changing a little bit the values $K(a_1)$ and $K(a_2)$ (see fig 1).

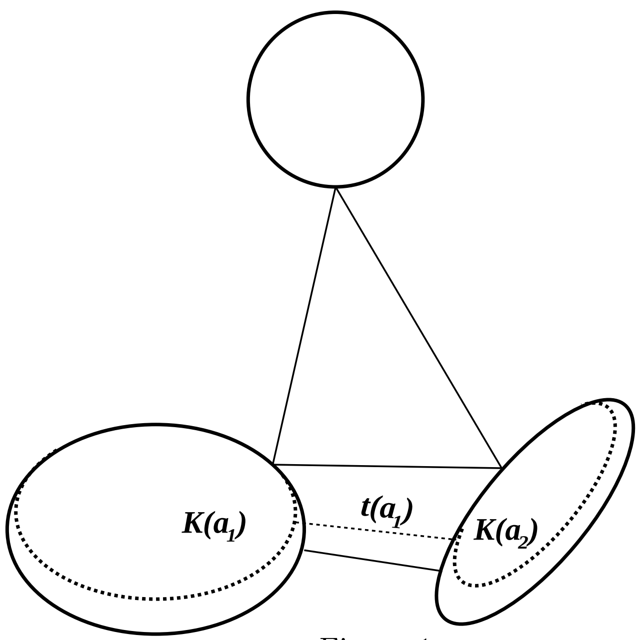

Figure 1

We will show that these changes will indeed change the value

$$((1 + t(a_1)k(a_1))(1 + t(a_2)k(a_2))).$$

Denote $t = t(a_1) = t(a_2) = t(a_3) = \cdots$, $k_1 = k(a_1)$ and $k_2 = k(a_2)$. Suppose that with the above described changes the value $(1 + tk_1)(1 + tk_2)$ remains constant equal to d.

The first equation we consider is

$$(1 + tk_1)(1 + tk_2) = d. \tag{3}$$

Note that

$$c_1 = 2K(a_1) = c_1(a_1), c_5(a_1), c_9(a_1), \ldots,$$

$$c_2 = 2K(a_2) = c_3(a_1), c_7(a_1), c_{11}(a_1), \ldots$$

and, for all $k \in \mathbf{N}$

$$c_{2k} = t.$$

Note also that $c_3(a_1) = c_1(a_2)$, etc...
Therefore,

$$k_1 = k(a_1) = c_1(a_1) + \cfrac{1}{c_2(a_1) + \cfrac{1}{c_1(a_2) + \cfrac{1}{c_2(a_2) + \cdots}}} = c_1 + \cfrac{1}{t + \cfrac{1}{k_2}},$$

and we obtain from this last expression our second equation

$$tc_1k_2 + c_1 + k_2 - tk_1k_2 - k_1 = 0, \tag{4}$$

So, k_1 and k_2 clearly depend continuously on c_1, c_2 and t.
From (3),

$$t(k_1 + k_2) + t^2k_1k_2 = d - 1. \tag{5}$$

Multiplying (4) by t one obtains

$$t^2c_1k_2 + c_1t + k_2t = tk_1 + t^2k_1k_2,$$

and now adding tk_2 to both members of last expression, one obtains from (5) that

$$t^2c_1k_2 + c_1t + 2k_2t = tk_1 + tk_2 + t^2k_1k_2 = d - 1.$$

The expression

$$t(tc_1k_2 + c_1 + 2k_2) = d - 1 \tag{6}$$

shows that k_2 depends only on t and c_1. Note that in changing $K(a_1)$ (respectively $K(a_2)$) we also change $c_1 = \frac{2K(a_1)}{\cos\phi(a_1)} = 2K(a_1)$ (respectively c_2).

Now, from the periodicity of a_2

$$k_2 = k(a_2) = c_1(a_2) + \cfrac{1}{c_2(a_2) + \cfrac{1}{c_3(a_2) + \cfrac{1}{c_4(a_2) + \cfrac{1}{k_2 + \cdots}}}}$$

and finally

$$k_2 = c_2 + \cfrac{1}{t + \cfrac{1}{c_1 + \cfrac{1}{t + \cfrac{1}{k_2 + \cdots}}}}. \tag{7}$$

The last expression shows that k_2 depends on c_2, c_1 and t (in fact is a solution of a quadratic equation whose coefficients depend on c_1, c_2, t). Note from (7) that k_2 really changes with the value c_2, that is, for t, c_1 fixed, k_2 depends on c_2. If (3) is true, then (6) says that k_2 is constant for t, c_1 fixed.

The conclusion is that the assumption (3) with d constant is false. Therefore we are able to perturb $(O_1, O_2, O_3) \in B_{m_1,n_1}$ obtaining that (2) is not true anymore. Thus, each set B_{m_1,n_1} is nowhere dense, and therefore by the Baire Category Theorem, for a dense G_δ set of (O_1, O_2, O_3) in F, equation (1) is not true for any m_1, n_1. Therefore, the potential ψ is non-lattice and the proof of Theorem 1 is complete.

Bibliography:

1) A. Lopes and R. Markarian, *Open billiards: invariant and conditionally invariant probabilities on Cantor sets*, to appear in SIAM Jour. of Appl. Math.

2) T. Morita, *The symbolic representation of billiards without boundary conditions*, Trans A.M.S., Vol 325, (1991) pp 819-828

3) W. Parry and M. Pollicott, *Zeta functions and the periodic structure of hyperbolic dynamics*, Asterisque 187-188 (1990)

4) C. Robinson, *Dynamical Systems*, CRC Press, 1995

5) Y. Sinai, *Dynamical Systems with elastic reflections*, Russian Math. Surveys, 25:1 (1970) pp 137-189